Missouri S&T Library and Learning Resources

Scholars' Mine

Doctoral Dissertations | Student Theses and Dissertations

1973

An application of residual relaxation to mathematical simulation of petroleum reservoirs

Grufron Achmad

Follow this and additional works at: https://scholarsmine.mst.edu/doctoral_dissertations

 Part of the Petroleum Engineering Commons

Department: Geosciences and Geological and Petroleum Engineering

Recommended Citation

Achmad, Grufron, "An application of residual relaxation to mathematical simulation of petroleum reservoirs" (1973). *Doctoral Dissertations*. 234.
https://scholarsmine.mst.edu/doctoral_dissertations/234

This thesis is brought to you by Scholars' Mine, a service of the Missouri S&T Library and Learning Resources. This work is protected by U. S. Copyright Law. Unauthorized use including reproduction for redistribution requires the permission of the copyright holder. For more information, please contact scholarsmine@mst.edu.

AN APPLICATION OF RESIDUAL RELAXATION TO MATHEMATICAL SIMULATION OF PETROLEUM RESERVOIRS

by

GRUFRON ACHMAD, 1935-

A DISSERTATION

Presented to the Faculty of the Graduate School of the

UNIVERSITY OF MISSOURI-ROLLA

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

PETROLEUM ENGINEERING

1973

T2808
92 pages
c.1

M.D. Arnold
Advisor

J.P. Gonier

A. Hester Haney

R. Boarelle

C.Y. Ho

J.C. Wilson

237286

ABSTRACT

Stability and convergence have long been primary concerns in the application of numerical techniques to the simulation of petroleum reservoir performance. An iterative method based on sequentially reducing the residual to zero at each grid point has been developed. It has been shown analytically that this residual relaxation method is stable, convergent, and consistent.

A three-phase areal model was developed to simulate a hypothetical homogeneous petroleum reservoir, and various computational methods were tested with this model. The residual relaxation method developed in this study (RELAX) was found to converge as rapidly as SIP, and much faster than either LSOR or LSOR+2DC. The RELAX+ADIP combination was found to converge more rapidly than SIP. No iteration parameter is needed for either RELAX or RELAX+ADIP, and both methods are easily programmed.

The RELAX+ADIP combination was found to be the best method tested for solving the pressure equation in the simulation model.

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Dr. M. D. Arnold, his advisor, without whose tireless help and understanding this work would not have been possible.

The author also wishes to express his appreciation to Dr. A. H. Harvey and Dr. C. Y. Ho for their careful and thorough review of this dissertation.

The author also wishes to express his gratitude to Prof. J. P. Govier and Dr. R. E. Carlile for having served on the Graduate Committee and critically examined this dissertation.

Special thanks are due to UNESCO for granting a scholarship to the author during the period January 1969 to December 1970. Special thanks are also due to the Asia Foundation for granting a scholarship to the author during the period January 1971 to December 1972.

TABLE OF CONTENTS

Table of Contents (continued)

LIST OF FIGURES

LIST OF TABLES

I. INTRODUCTION

The use of mathematical simulation of reservoir fluid flow to predict field performance has been widely accepted by the petroleum industry. The multidimensional and multiphase flow of compressible fluids is correctly described by non-linear partial differential equations. The solution to the differential equations by means of finite difference approximation relies upon reducing the equations to algebraic systems. Depending upon how the non-linear coefficients and the unknown variables are treated, various sets of equations result. A number of methods for solving these equations based on iterative and non-iterative techniques have been developed.

Stability and convergence conditions inherent in the calculation process have always been a prime concern in the application of these techniques. A method is considered efficient if it would be stable and converge in the acceptable range of reservoir geometry, time increment, production rate, and transmissibility. Numerous authors have made studies to investigate the efficiency of each method. The relevant conclusions derived from their investigations were that iterative methods are superior to the non-iterative ones, and among the iterative methods the most favorable ones are the Strongly Implicit Procedure (SIP), the Alternating Direction Implicit Procedure (ADIP), and the Successive Over-relaxation Method (SOR). These iterative techniques require that an iteration parameter be determined. Despite the formulas that are available to calculate this parameter, the

maximum iteration parameter must be determined by a trial and error solution, which is in most cases quite cumbersome.

The purpose of this study is to secure stability in solving the pressure equation. A method based on forcing the residuals to zero at each grid point of each iterative step has been developed in this study; the residual relaxation scheme is presented. The favorable features of this residual relaxation method are: it is stable, it converges rapidly, it is not difficult to program, and it is useful for solving problems involving parabolic as well as elliptic types of partial differential equations.

In order to test the feasibility of the method, a three-phase areal model was developed to simulate black oil reservoir performance. A complete description of the model is presented in Appendices A, B, and C.

II. LITERATURE REVIEW

Since its introduction in 1955 by Peaceman and Rachford,[1] the ADIP method has frequently been used as the standard of comparison for testing other methods employed for reservoir simulation. In their investigation Peaceman and Rachford proved that ADIP was unconditionally stable, and that it converged more rapidly than the extrapolated Liebman method.

Quon, Dranchuk, Allada, and Leung in 1965,[2] and later Coats and Terhune in 1966,[3] made studies of the Alternating Direction Explicit Procedure (ADEP), and compared it to ADIP. They observed that ADEP required less computational effort, but that the accuracy of the results was less than could be obtained by the ADIP technique.

Various types of SOR methods were studied by Bjordammen and Coats in 1967;[4] and later in 1968, Briggs and Dixon[5] made studies of some of the SOR methods. The results of these investigations were that ADIP was superior to all types of SOR methods with a single iteration parameter, and that the Line Successive Over-relaxation method (LSOR) was efficient in solving problems involving heterogeneous reservoirs where a small number of grids were suitable for the simulation. The closure criteria used by Bjordammen and Coats in these studies were that the maximum sum of residuals, the material balance error, and maximum saturation change in the last iteration must not exceed specified tolerances.

Stone in 1968[6] made a comparative study of SIP, ADIP, and LSOR. The results of this investigation were that SIP converged faster than

the other methods for increased heterogeneity of the reservoir, while ADIP converged more rapidly for homogeneous reservoirs simulated with square grids. Later, in 1970, Weinstein, Stone, and Kwan,[7] and again in 1971, Steen and Ali,[8] reported obtaining similar results. However, Traylor and Sheffield in 1971[9] added the conclusions that SIP was sensitive to the initial pressure estimate, and that ADIP was sensitive to the dimension of the grid and homogeneity of the reservoir.

Watts in 1971[10] added a correction factor to LSOR. This correction factor was calculated for each row by forcing the sum of the residuals of that row to zero. The purpose of this correction was to remove the components of certain eigenvectors from the solution vectors, so that the rate of convergence was increased. Watts indicated that the corrected LSOR was very effective for strongly anisotropic problems in comparison to SIP and ADIP. Later in 1972, Aziz and Settari[11] corrected LSOR in two dimensions. The results of this investigation were that SIP and their method, LSOR+2DC, were the only two methods that converged for all problems tested, and that ADIP was competitive with SIP for elliptic cases; also, 2DC could be easily programmed.

Rout and Crawford in 1973[12] developed a new method which was based on calculating the five unknown pressures at the grid points (i,j), (i-1,j), (i+1,j), (i,j-1) and (i,j+1), from the known eight pressures surrounding them. According to their investigation, this new method always converged, and in terms of computation time required, the new method was competitive to SIP only for small size problems.

It can be summarized from the above-mentioned studies, that the

most favorable methods are SIP, ADIP, and LSOR. Each of these methods is most efficient in solving particular types of problems. LSOR gives better results for small anisotropic problems; ADIP is more efficient for simulating homogeneous reservoirs and for solving elliptic equations; SIP is best for solving problems having increased heterogeneity and anisotropy.

The relaxation method was first introduced by Southwell in 1935.[13] Since then many studies have applied the relaxation technique for the solution of linear problems in various fields. Most of these studies have been conducted in Great Britain.

Temple in 1939[14] developed a general theory of relaxation in terms of the theory of linear operators and further introduced a method of steepest descent.

Emmons in 1943[15] applied the relaxation technique in solving multidimensional steady-state flow heat conduction problems. He recognized that the relaxation method was far superior to the analytical method of solution in terms of time required to reach a desired accuracy.

Dykstra and Parsons in 1951[16] introduced the relaxation method to oil field research, in order to calculate the true pressure drop measured in the Hassler type relative permeability apparatus. They came to the same conclusions as Emmons, that practically any problem of steady-state condition can be solved by the relaxation method and that the solution can be obtained in a relatively shorter time period in comparison to the analytical method.

Mitchell and Rutherford in 1953[17] illustrated analytically that

the numerical solution of the relaxation technique converged to the solution of the finite difference equations.

Allen in 1954[18] demonstrated how to develop relaxation schemes for the solution of various linear partial differential equations. He indicated that the relaxation method always converged to a solution.

Scarborough in 1955[19] solved Laplace's equation by a relaxation method, and indicated that solutions with a desired accuracy could be obtained, even though an arithmetic error had been introduced during the process of calculation. This illustrated the stability of the relaxation method.

The general procedure of the relaxation method, as initially suggested by Southwell, was to relax first the largest residual, followed by the next largest one, and to continue in this manner until all the residuals of each point in the system were liquidated. If the desired accuracy was not reached, this process of calculation was repeated. Because of this particular characteristic in the process of calculation the relaxation method was best performed by using a large piece of paper, a pencil, and a hand calculator.

The author has developed a relaxation scheme adaptable for computerized calculation for the solution of the non-linear pressure equation. The scheme is presented in the next chapter.

III. THEORETICAL ANALYSIS

When a numerical method is employed in obtaining a solution of a differential equation, the process involves two types of approximations. The exact solution, u, of the finite difference equation is, at best, only an approximation to the true solution, U, of the partial differential equation. The convergence of u to U, as the mesh size approaches zero, depends upon the finite difference approximation employed.

The second approximation is in obtaining the value u of the finite difference equation by the numerical method used. If it were possible to carry out all the calculations in the numerical method to an infinite number of decimal places, the exact solution, u, of the finite difference solution would be obtained. In practice, however, each calculation will produce a round-off error, and the solution computed will be N instead of u. The numerical method is stable when the error decays in the process of calculation.

A. Convergence of the Implicit Finite Difference Equation

The difference between the true solution of the differential equation and the exact solution of its corresponding difference equation, $(U - u)$, is called the discretization error. For a two-dimensional problem, the magnitude of the discretization error at any grid point depends on the sizes of Δx, Δy, and Δt. In general, by decreasing Δx, Δy, and Δt, the discretization error can be made small, and upper and lower bounds can be established. In such a case the finite difference

approximation converges to the solution of the partial differential equation.

The analytical treatment of convergence of the implicit type of the finite difference equation presented here follows that given by Lowan[20] and Smith[21] for an explicit type of finite difference equation.

Denote $e = U - u$. (III-1)

Consider the differential equation

$$\frac{\partial U}{\partial t} = \frac{k_x}{\phi \mu c} \frac{\partial^2 U}{\partial x^2} + \frac{k_y}{\phi \mu c} \frac{\partial^2 U}{\partial y^2}. \tag{III-2}$$

The implicit finite difference approximation to (III-2) is

$$u_{i,j}^{n+1} - u_{i,j}^{n} = \eta_x u_{i-1,j}^{n+1} - 2\eta_x u_{i,j}^{n+1} + \eta_x u_{i+1,j}^{n+1} +$$

$$\eta_y u_{i,j-1}^{n+1} - 2\eta_y u_{i,j}^{n+1} + \eta_y u_{j+1}^{n+1} , \tag{III-3}$$

where

$$\eta_x = \frac{k_x}{\phi \mu c} \frac{\Delta t}{\Delta x^2} , \text{ and } \eta_y = \frac{k_y}{\phi \mu c} \frac{\Delta t}{\Delta y^2} .$$

For each grid point

$$u_{i,j}^{n} = U_{i,j}^{n} - e_{i,j}^{n}; \text{ and } u_{i,j}^{n+1} = U_{i,j}^{n+1} - e_{i,j}^{n+1} .$$

Substitution in (III-3) leads to

$$e^{n+1}_{i,j} - e^{n}_{i,j} = \eta_x e^{n+1}_{i-1,j} - 2\eta_x e^{n+1}_{i,j} + \eta_x e^{n+1}_{i+1,j} + \eta_y e^{n+1}_{i,j-1}$$

$$- 2\eta_y e^{n+1}_{i,j} + \eta_y e^{n+1}_{i,j+1} + U^{n+1}_{i,j} - U^{n}_{i,j}$$

$$- \eta_x U^{n+1}_{i-1,j} + 2\eta_x U^{n+1}_{i,j} - \eta_x U^{n+1}_{i+1,j} - \eta_y U^{n+1}_{i,j-1}$$

$$+ 2\eta_y U^{n+1}_{i,j} - \eta_y U^{n+1}_{i,j+1} \; . \qquad \text{(III-4)}$$

The Taylor's series expansion of the implicit terms gives:

$$U^{n+1}_{i,j} = U^{n}_{i,j} + \Delta t \, \frac{\partial U(x_i, y_j, t^n + \theta_o \, \Delta t)}{\partial t}$$

also $$U^{n+1}_{i,j} = U^{n}_{i,j} + \Delta t \left(\frac{\partial U}{\partial t}\right)^{n}_{i,j} + \frac{\Delta t^2}{2!} \, \frac{\partial^2 U(x_i, y_j, t^n + \theta_1 \Delta t)}{\partial t^2} \; ,$$

and $$U^{n+1}_{i-1,j} = U^{n}_{i,j} - \Delta x \left(\frac{\partial U}{\partial x}\right)^{n}_{i,j} + \Delta t \left(\frac{\partial U}{\partial t}\right)^{n}_{i,j} +$$

$$\frac{\Delta x^2}{2!} \, \frac{\partial^2 U(x_i - \theta_2 \Delta x, y_j, t^n + \theta_1 \Delta t)}{\partial x^2} -$$

$$\frac{2 \Delta x \Delta t}{2!} \, \frac{\partial^2 U(x_i - \theta_2 \Delta x, y_j, t^n + \theta_1 \Delta t)}{\partial x \partial t} +$$

$$\frac{\Delta t^2}{2!} \, \frac{\partial^2 U(x_i - \theta_2 \Delta x, y_j, t^n + \theta_1 \Delta t)}{\partial t^2} \; ,$$

similarly,

$$U^{n+1}_{i+1,j} = U^{n}_{i,j} + \Delta x \left(\frac{\partial U}{\partial x}\right)^{n}_{i,j} + \Delta t \left(\frac{\partial U}{\partial x}\right)^{n}_{i,j} +$$

$$\frac{\Delta x^2}{2!}\frac{\partial^2 U(x_i + \theta_2\Delta x, y_j, t^n + \theta_1\Delta t)}{\partial x^2} +$$

$$\frac{2\Delta x\Delta t}{2!}\frac{\partial^2 U(x_i + \theta_2\Delta x, y_j, t^n + \theta_1\Delta t)}{\partial x\partial t} +$$

$$\frac{\Delta t^2}{2!}\frac{\partial^2 U(x_i + \theta_2\Delta x, y_j, t^n + \theta_1\Delta t)}{\partial t^2},$$

and $$U^{n+1}_{i,j-1} = U^n_{i,j} - \Delta y(\frac{\partial U}{\partial y})^n_{i,j} + \Delta t(\frac{\partial U}{\partial t})^n_{i,j} +$$

$$\frac{\Delta y^2}{2!}\frac{\partial^2 U(x_i, y_j - \theta_3\Delta y, t^n + \theta_1\Delta t)}{\partial y^2} -$$

$$\frac{2\Delta y\Delta t}{2!}\frac{\partial^2 U(x_i, y_j - \theta_3\Delta y, t^n + \theta_1\Delta t)}{\partial y\partial t} +$$

$$\frac{\Delta t^2}{2!}\frac{\partial^2 U(x_i, y_j - \theta_3\Delta y, t^n + \theta_1\Delta t)}{\partial t^2},$$

similarly,

$$U^{n+1}_{i,j+1} = U^n_{i,j} + \Delta y(\frac{\partial U}{\partial y})^n_{i,j} + \Delta t(\frac{\partial U}{\partial t})^n_{i,j} +$$

$$\frac{\Delta y^2}{2!}\frac{\partial^2 U(x_i, y_j + \theta_3\Delta y, t^n + \theta_1\Delta t)}{\partial y^2} +$$

$$\frac{2\Delta y\Delta t}{2!}\frac{\partial^2 U(x_i, y_j + \theta_3\Delta y, t^n + \theta_1\Delta t)}{\partial y\partial t} +$$

$$\frac{\Delta t^2}{2!}\frac{\partial^2 U(x_i, y_j + \theta_3\Delta y, t^n + \theta_1\Delta t)}{\partial t^2}.$$

Substitution into (III-4) results in

$$e_{i,j}^{n+1} - e_{i,j}^{n} = \eta_x e_{i-1,j}^{n+1} - 2\eta_x e_{i,j}^{n+1} + \eta_x e_{i+1,j}^{n+1} +$$

$$\eta_y e_{i,j-1}^{n+1} - 2\eta_y e_{i,j}^{n+1} + \eta_y e_{i,j+1}^{n+1} +$$

$$\Delta t\{\frac{\partial U(x_i,y_j,t^n + \theta_o\Delta t)}{\partial t} -$$

$$\frac{1}{\phi\mu c}\left(k_x \frac{\partial^2 U(x_i + \theta_4\Delta x, y_j, t^n + \theta_1\Delta t)}{\partial x^2} +\right.$$

$$\left. k_y \frac{\partial^2 U(x_i, y_j + \theta_5\Delta y, t^n + \theta_1\Delta t)}{\partial y^2}\right)\}. \qquad \text{(III-5)}$$

Let E^n denote the modulus of the maximum error along the nth time step; then equation (III-5) can be written as

$$E^{n+1} - E^n = \Delta t\{\frac{\partial U(x_i,y_j,t^n + \theta_o\Delta t)}{\partial t} - \frac{1}{\phi\mu c}\left(k_x\right.$$

$$\frac{\partial^2 U(x_i + \theta_4\Delta x, y_j, t^n + \theta_1\Delta t)}{\partial x^2} +$$

$$\left. k_y \frac{\partial^2 U(x_i, y_j + \theta_5\Delta y, t^n + \theta_1\Delta t)}{\partial y^2}\right)\}. \qquad \text{(III-6)}$$

Let M be the maximum value of the expression in the brackets; then (III-6) becomes

$$E^{n+1} \leq E^n + (\Delta t)M,$$

or $$E^{n+1} \leq E^{n-1} + 2(\Delta t)M,$$

or $$E^{n+1} \leq E^o + n(\Delta t)M,$$

or $$E^{n+1} \leq n(\Delta t)M,$$

because $E^o = 0$; that is, the initial values of u and U are the same. When Δt tends to zero, then M approaches to the solution of

$$\frac{\partial U}{\partial t} - \frac{1}{\phi \mu c} \left(k_x \frac{\partial^2 U}{\partial x^2} + k_y \frac{\partial^2 U}{\partial y^2} \right) .$$

Since U is the solution of (III-2), the limiting value of M and therefor of E^n is zero. E^n is the modulus of the maximum error, therefore $|U - u| < E^n$. In this case u converges to U as Δt tends to zero.

B. Convergence and Stability of Residual Relaxation Method

The residual relaxation method is an iterative scheme for obtaining a numerical solution N of the finite difference equation. This method is one which forces N to approach u. Consequently, the method will be stable. The analytical treatment of the residual relaxation method presented here is based on the works done by Temple, and by Mitchell and Rutherford.

Consider a finite difference equation which can be written in the matrix form of

$$Ax = D, \qquad \text{(III-7)}$$

where A is the coefficient matrix, x is the solution vector, and D is a column vector which is known.

Let x^1 be a trial solution, and set

$$Ax^1 - D = R^1. \qquad \text{(III-8)}$$

Then R is a residual column vector.

If a column vector y^o, y^1, y^2,...,y^n can be found such that R^o, R^1, R^2,..., R^n defined by

$$\begin{aligned} R^o - Ay^o &= R^1, \\ R^1 - Ay^1 &= R^2, \\ &\vdots \\ R^{n-1} - Ay^{n-1} &= R^n, \end{aligned}$$

tend monotonically to zero, then the vector

$$x^o,\ x^1 = x^o - y^o,\ x^2 = x^1 - y^1,\ldots,\ x^n = x^{n-1} - y^{n-1},$$

tends to the solution x of Ax = D; because

$$\begin{aligned} Ax^1 - D &= A(x^o - y^o) - D = R^o + R^1 - R^o = R^1, \\ Ax^2 - D &= A(x^1 - y^1) - D = R^1 + R^2 - R^1 = R^2, \\ &\vdots \\ Ax^n - D &= A(x^{n-1} - y^{n-1}) - D = R^{n-1} + R^n - R^{n-1} = R^n. \end{aligned}$$

The vector y^n is chosen to have only one non-zero element; assume the jth row, therefore

$$y_r^n = \delta_{rj} k_j, \qquad \text{(III-9)}$$

where δ_{rj} is Kronecker delta.

The component R_r^{n+1} of the residual vector R^{n+1} will be

$$R_r^{n+1} = R_r^n - a_{rj} k_j^n. \qquad \text{(III-10)}$$

The value of k_j^n can be determined by studying the quadratic form of equation (III-10)

$$(R_r^{n+1})^2 = (R_r^n - a_{rj}k_j^n)^2,$$

and
$$|R_r^{n+1}|^2 = |R_r^n|^2 - (2a_{rj}k_j^nR_r^n - a_{rj}^2k_j^{n2}). \qquad \text{(III-11)}$$

Thus
$$|R_r^{n+1}| < |R_r^n|,$$

if
$$2a_{rj}k_j^nR_r^n - a_{rj}^2k_j^{n2} > 0. \qquad \text{(III-12)}$$

The left hand side of (III-12) will be maximum for

$$k_j^n = R_r^n/a_{rj}, \qquad \text{(III-13)}$$

and the maximum value is

$$(R_r^n)^2,$$

or it can be written as

$$k_j^{n2}\ a_{rj}^2\ . \qquad \text{(III-14)}$$

The discussion above has shown that by assigning $y_j^n = R_j^n/a_{jj}$, an improvement on the value of x^{n+1} can be effected such that $|R^{n+1}| < |R^n|$. Next it will be shown, that by choosing the appropriate value of y, R decreases monotonically to zero.

Assume that R tends to a lower limit L. For any $\varepsilon > 0$, there will be n, such that

$$|R^n|^2 - L^2 < \varepsilon.$$

But it is possible to reduce $|R^n|^2$ by ε (refer to (III-14)), unless

$$(k_j^n)^2(a_{rj})^2 < \varepsilon,$$

or, unless $$(k_j^n)^2 < \varepsilon/(a_{rj})^2,$$

or, unless $$L^2 \leq |R^n|^2 < \varepsilon;$$

in this case $L = 0$, and $\lim_{n\to\infty} |R^n| = 0$.
Therefore the numerical solution N converges to the finite difference approximation u.

The relaxation process consists of reducing the residuals from whatever the original values are to zero by repeatedly applying equation (III-10), which is

$$R_r^{n+1} = R_r^n - a_{rj}k_j.$$

Suppose errors are introduced in the process of relaxing the residuals, such that instead of R_r^n, R_r^{n*} was calculated.

Assign the error vector

$$e_r^n = R_r^{n*} - R_r^n. \qquad \text{(III-15)}$$

The error vector for the next (n+1) step will then be e_r^{n+1}. The residuals are:

$$R_r^{n+1} = R_r^n - a_{rj}k_j^n ,$$

or $$R_r^{n+1} = R_r^n - a_{rj}(R_j^n/a_{jj}), \qquad \text{(III-16)}$$

and $$R_r^{n+1*} = R_r^{n*} - a_{rj}(R_j^{n*}/a_{jj}). \qquad \text{(III-17)}$$

Subtracting equation (III-16) from equation (III-17) yields

$$e_r^{n+1} = e_r^n - a_{rj}(e_j^n/a_{jj}). \qquad \text{(III-18)}$$

Equation (III-18) is exactly the same recursion formula as equation (III-10), except that (III-18) is written for the error vector e. Analogous to the above discussion on the residual vector, the following will be true for the error vector; that is

$$|e^{n+1}| < |e^{n}|,$$

and

$$\lim_{n\to\infty} |e^{n}| = 0.$$

Therefore the residual relaxation scheme is unconditionally stable.

In order to increase the rate of convergence, in reference to equation (III-13), the value of k_j^n must be maximum. In the original relaxation scheme, Southwell made k_j^n maximum by selecting R_r maximum. Mitchell and Rutherford modified k_j^n by

$$k_j^n = (\sum_r R_r^n a_{rj})/(\sum_r a_{rj}^2);$$

they chose to relax first the residual with the highest k_j^n.

The above discussion has dealt with a parabolic equation. In the case of an elliptic equation, vector D in equation (III-7) is a zero vector. In the process of calculating the residual vector R, D is on the right hand side. In the elliptic case the residual vector R will then be greater by a magnitude of D than in the parabolic case. All else being the same, the relaxation scheme for the elliptic case will then be similar to the parabolic case. Therefore the relaxation scheme will also be stable for elliptic type of problems.

C. Compatibility

It has been shown that the implicit difference approximation converges, and it has also been shown that the relaxation scheme converges to the finite difference approximation and that it is stable. In this section it will be shown that the implicit difference approximation is compatible or consistent with the differential equation which it represents. The proof will follow the method used by Smith for the explicit difference scheme.

Compatibility is measured in terms of the difference between the differential equation and the implicit finite difference equation. This difference is called the truncation error and is denoted by T. Consider the partial differential equation

$$\frac{\partial u}{\partial t} - \eta_x \frac{\partial^2 u}{\partial x^2} - \eta_y \frac{\partial^2 u}{\partial y^2} = 0 , \qquad \text{(III-19)}$$

where $\eta_x = k_x/\phi\mu c$, and $\eta_y = k_y/\phi\mu c$.

The implicit finite difference of equation (III-19) is

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} - \eta_x \frac{u_{i-1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j}^{n+1}}{\Delta x^2} - \eta_y \frac{u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}}{\Delta y^2} = 0.$$

Let v(x,y,t) be a function possessing continuous partial derivatives. Then the truncation error T is

$$\left(\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} - \eta_x \frac{v_{i-1,j}^{n+1} - 2v_{i,j}^{n+1} + v_{i+1,j}^{n+1}}{\Delta x^2} - \eta_y \frac{v_{i,j-1}^{n+1} - 2v_{i,j}^{n+1} + v_{i,j+1}^{n+1}}{\Delta y^2}\right)$$

$$- \left(\frac{\partial v}{\partial t} - \eta_x \frac{\partial^2 v}{\partial x^2} - \eta_y \frac{\partial^2 v}{\partial y^2}\right) = T. \qquad \text{(III-20)}$$

When $v = U$, where U is the exact solution of the differential equation (III-19), then

$$\frac{\partial U}{\partial t} - \eta_x \frac{\partial^2 U}{\partial x^2} - \eta_y \frac{\partial^2 U}{\partial y^2} = 0.$$

Hence the expansion of T provides a measure of the rate at which the value of the finite difference equation approaches the value of the differential equation.

The Taylor's series expansions of the finite difference terms are:

$$v_{i,j}^{n+1} = v_{i,j}^{n} + \Delta t \frac{\partial v}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 v}{\partial t^2} + \frac{\Delta t^3}{3!} \frac{\partial^3 v}{\partial t^3} \quad . \quad . \quad . \quad ,$$

$$v_{i-1,j}^{n+1} = v_{i,j}^{n} - \Delta x \frac{\partial v}{\partial x} + \Delta t \frac{\partial v}{\partial t} + \frac{\Delta x^2}{2!} \frac{\partial^2 v}{\partial x^2}$$

$$- \frac{2\Delta x \Delta t}{2!} \frac{\partial^2 v}{\partial x \partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 v}{\partial t^2}$$

$$- \frac{\Delta x^3}{3!} \frac{\partial^3 v}{\partial x^3} + \frac{2\Delta x^2 \Delta t}{3!} \frac{\partial^3 v}{\partial x^2 \partial t} - \frac{\Delta x \Delta t^2}{3!} \frac{\partial^3 v}{\partial x \partial t^2}$$

$$+ \frac{\Delta x^2 \Delta t}{3!} \frac{\partial^3 v}{\partial x^2 \partial t} - \frac{2\Delta x \Delta t^2}{3!} \frac{\partial^3 v}{\partial x \partial t^2} + \frac{\Delta t^3}{3!} \frac{\partial^3 v}{\partial t^3}$$

$$+ \frac{\Delta x^4}{4!} \frac{\partial^4 v}{\partial x^4} - \frac{3\Delta x^3 \Delta t}{4!} \frac{\partial^4 v}{\partial x^3 \partial t} + \frac{3\Delta x^2 \Delta t^2}{4!} \frac{\partial^4 v}{\partial x^2 \partial t^2}$$

$$- \frac{\Delta x \Delta t^3}{4!} \frac{\partial^4 v}{\partial x \partial t^3} - \frac{\Delta x^3 \Delta t}{4!} \frac{\partial^4 v}{\partial x^3 \partial t} + \frac{3\Delta x^2 \Delta t^2}{4!} \frac{\partial^4 v}{\partial x^2 \partial t^2}$$

$$- \frac{3\Delta x \Delta t^3}{4!} \frac{\partial^4 v}{\partial x \partial t^3} + \frac{\Delta t^4}{4!} \frac{\partial^4 v}{\partial t^4} \quad . \quad . \quad . \quad ,$$

$$v_{i+1,j}^{n+1} = v_{i,j}^{n} + \Delta x \frac{\partial v}{\partial x} + \Delta t \frac{\partial v}{\partial t} + \frac{\Delta x^2}{2!} \frac{\partial^2 v}{\partial x^2} + \frac{2\Delta x \Delta t}{2!} \frac{\partial^2 v}{\partial x \partial t}$$

$$+ \frac{\Delta t^2}{2!} \frac{\partial^2 v}{\partial t^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 v}{\partial x^3} + \frac{2\Delta x^2 \Delta t}{3!} \frac{\partial^3 v}{\partial x^2 \partial t} + \frac{\Delta x \Delta t^2}{3!} \frac{\partial^3 v}{\partial x \partial t^2}$$

$$+ \frac{\Delta x^2 \Delta t}{3!} \frac{\partial^3 v}{\partial x^2 \partial t} + \frac{2\Delta x \Delta t^2}{3!} \frac{\partial^3 v}{\partial x \partial t^2} + \frac{\Delta t^3}{3!} \frac{\partial^3 v}{\partial t^3}$$

$$+ \frac{\Delta x^4}{4!} \frac{\partial^4 v}{\partial x^4} + \frac{3\Delta x^3 \Delta t}{4!} \frac{\partial^4 v}{\partial x^3 \partial t} + \frac{3\Delta x^2 \Delta t^2}{4!} \frac{\partial^4 v}{\partial x^2 \partial t^2}$$

$$+ \frac{\Delta x \Delta t^3}{4!} \frac{\partial^4 v}{\partial x \partial t^3} + \frac{\Delta x^3 \Delta t}{4!} \frac{\partial^4 v}{\partial x^3 \partial t} + \frac{3\Delta x^2 \Delta t^2}{4!} \frac{\partial^4 v}{\partial x^2 \partial t^2}$$

$$+ \frac{3\Delta x \Delta t^3}{4!} \frac{\partial^4 v}{\partial x \partial t^3} + \frac{\Delta t^4}{4!} \frac{\partial^4 v}{\partial t^4} \cdots \quad .$$

Similarly for $v_{i,j-1}^{n+1}$, and $v_{i,j+1}^{n+1}$.

Substitution in the equation (III-20) gives

$$T = \frac{\Delta t}{2!} \frac{\partial^2 v}{\partial t^2} + \eta_x \frac{(2)3\Delta t}{3!} \frac{\partial^3 v}{\partial x^2 \partial t} + \eta_x \frac{2\Delta x^2}{4!} \frac{\partial^4 v}{\partial x^4}$$

$$+ \eta_y \frac{(2)3\Delta t}{3!} \frac{\partial^3 v}{\partial y^2 \partial t} + \eta_y \frac{2\Delta y^2}{4!} \frac{\partial^4 v}{\partial y^4} \, .$$

$$T = O(\Delta t) + O(\Delta x^2) + O(\Delta y^2).$$

Thus as Δt, Δx, and Δy approach zero, T also approaches zero. Therefore the implicit difference equation is consistent with the partial differential equation.

IV. CALCULATION SCHEME

The calculation scheme of the residual relaxation method will be shown in this chapter. The pressure equation of the implicit case can be expressed in its compact matrix form as

$$AP^{n+1} = D \quad . \tag{IV-1}$$

Transferring the D term to the left, equation (IV-1) can be written as

$$AP^{n+1} - D = 0. \tag{IV-2}$$

The right side of equation (IV-2) will only be zero if P^{n+1} is the exact solution to the equation (IV-1). For any arbitrary value of P^{n+1}, the right side of equation (IV-2) will have a residual value. The residual may be written as

$$R = AP^{n+1} - D. \tag{IV-3}$$

The basic concept of the residual relaxation technique is to force the value of the current residual to zero at each iteration. At the position of grid point (i,j) the equation (IV-1) can be written as

$$AX_{i,j}P^{n+1}_{i-1,j} + CX_{i,j}P^{n+1}_{i+1,j} + AY_{i,j}P^{n+1}_{i,j-1} + CY_{i,j}P^{n+1}_{i,j+1}$$

$$+ BB_{i,j}P^{n+1}_{i,j} = TERMS_{i,j} - \left(\frac{TRM}{\Delta t}\right)_{i,j}P^{n}_{i,j}, \tag{IV-4}$$

where $BB_{i,j} = \left(-AX_{i,j} - CX_{i,j} - AY_{i,j} - CY_{i,j} - \left(\frac{TRM}{\Delta t}\right)_{i,j}\right)$,

and $TERMS_{i,j} = QTERM_{i,j} + GTERM_{i,j} + PCTERM_{i,j}$.

The other coefficient terms are defined in Appendix A. The normalized residual will be

$$RESID_{i,j} = (AX_{i,j}P^{n+1}_{i-1,j} + CX_{i,j}P^{n+1}_{i+1,j} + AY_{i,j}P^{n+1}_{i,j-1} + CY_{i,j}P^{n+1}_{i,j+1}$$

$$- TERMS_{i,j} + (\frac{TRM}{\Delta t})_{i,j}P^{n}_{i,j} + BB_{i,j}P^{n+1}_{i,j}) \frac{1}{BB_{i,j}} . \qquad (IV\text{-}5)$$

Let k denote the current iteration level; then equation (IV-5) becomes

$$RESID^{k}_{i,j} = (AX_{i,j}P^{n+1,k+1}_{i-1,j} + CX_{i,j}P^{n+1,k}_{i+1,j} + AY_{i,j}P^{n+1,k+1}_{i,j-1} + CY_{i,j}P^{n+1,k}_{i,j+1}$$

$$- TERMS_{i,j} + (\frac{TRM}{\Delta t})_{i,j}P^{n}_{i,j}) \frac{1}{BB_{i,j}} + P^{n+1,k}_{i,j} . \qquad (IV\text{-}6)$$

The residual at the particular grid point (i,j) can be forced to zero by subtracting $RESID^{k}_{i,j}$ from both sides of equation (IV-6). Noting that the normalized residual defined by equation (IV-5) has the units of pressure, and applying the correction to $P^{n+1,k}_{i,j}$, the new corrected pressure is

$$P^{n+1,k+1}_{i,j} = P^{n+1,k}_{i,j} - RESID^{k}_{i,j}. \qquad (IV\text{-}7)$$

Correcting the pressure of grid point (i,j) will affect the residuals of the four adjacent points (i-1,j), (i+1,j), (i,j-1), and (i,j+1). The correction on the residuals of the adjacent grid points will proceed as follows. Consider grid point (i-1,j); the residual at iteration step k is

$$RESID^{k}_{i-1,j} = \frac{1}{BB_{i-1,j}} (AX_{i-1,j} P^{n+1,k+1}_{i-2,j} + CX_{i-1,j} P^{n+1,k}_{i,j}$$

$$+ AY_{i-1,j} P^{n+1,k+1}_{i-1,j-1} + CY_{i-1,j} P^{n+1,k}_{i-1,j+1} - TERMS_{i-1,j}$$

$$+ (\frac{TRM}{\Delta t})_{i-1,j} P^{n}_{i-1,j} + BB_{i-1,j} P^{n+1,k+1}_{i-1,j}). \qquad \text{(IV-8)}$$

Investigate the right side of equation (IV-8), equation (IV-7) is the correction to the term $P^{n+1,k}_{i,j}$. Therefore, applying the correction of equation (IV-7) to equation (IV-8) means updating the residual of grid point (i-1,j) to the (k+1) iteration level.

$$RESID^{k+1}_{i-1,j} = \frac{1}{BB_{i-1,j}} (AX_{i-1,j} P^{n+1,k+1}_{i-2,j} + CX_{i-1,j} P^{n+1,k+1}_{i,j}$$

$$+ AY_{i-1,j} P^{n+1,k+1}_{i-1,j-1} + CY_{i-1,j} P^{n+1,k}_{i-1,j+1} - TERMS_{i-1,j}$$

$$+ (\frac{TRM}{\Delta t})_{i-1,j} P^{n}_{i-1,j} + BB_{i-1,j} P^{n+1,k+1}_{i-1,j}),$$

or $$RESID^{k+1}_{i-1,j} = \frac{1}{BB_{i-1,j}} (AX_{i-1,j} P^{n+1,k+1}_{i-2,j} + CX_{i-1,j} (P^{n+1,k}_{i,j}$$

$$- RESID^{k}_{i,j}) + AY_{i-1,j} P^{n+1,k+1}_{i-1,j-1} + CY_{i-1,j} P^{n+1,k}_{i-1,j+1}$$

$$- TERMS_{i-1,j} + (\frac{TRM}{\Delta t})_{i-1,j} P^{n}_{i-1,j} + BB_{i-1,j} P^{n+1,k+1}_{i-1,j}),$$

or after rearranging the terms

$$RESID^{k+1}_{i-1,j} = RESID^{k}_{i-1,j} - \frac{CX_{i-1,j}}{BB_{i-1,j}} RESID^{k}_{i,j}. \qquad \text{(IV-9)}$$

Similarly, the corrected residuals for the other grid points can be obtained

$$RESID^{k+1}_{i+1,j} = RESID^{k}_{i+1,j} - \frac{AX_{i+1,j}}{BB_{i+1,j}} RESID^{k}_{i,j} ,$$

$$RESID^{k+1}_{i,j-1} = RESID^{k}_{i,j-1} - \frac{CY_{i,j-1}}{BB_{i,j-1}} RESID^{k}_{i,j} ,$$

$$RESID^{k+1}_{i,j+1} = RESID^{k}_{i,j+1} - \frac{AY_{i,j+1}}{BB_{i,j+1}} RESID^{k}_{i,j} .$$

The liquidation of the residuals must be performed for each block in the system. As the relaxation proceeds from one iteration level to the next, the residuals are reduced to an ever smaller magnitude. In this manner the solution pressures are improved and the improved pressures will then converge toward the solution of equation (IV-1), within the tolerance selected.

V. SIMULATION MODEL

A two-dimensional, three-phase areal reservoir simulation model was developed. The rock and fluid compressibilities, gravity effects, and oil-water capillary pressure were taken into account. The development of the model and a complete description of it are given in Appendices A, B, and C.

In the calculation procedure, the pressure is determined implicitly from equation (C-1), and then saturations are computed explicitly by equations (C-2), (C-3), and (C-4). The coefficients of equation (C-1) include terms which depend on both pressure and saturation. Since pressures and saturations change with time, the use of implicit coefficients will add significantly to the stability of the computation. Since these coefficients for the subsequent (n+1) time level are unknown, they must be estimated and updated iteratively. On the other hand, the linear nature of the solution will improve the rate of convergence of the computation. In order to maintain the implicit character of the coefficients while retaining the linear nature of the solution, this updating of the coefficients is accomplished by computing them on the basis of pressures and saturations which are extrapolated linearly from the prior time level. The linear prediction of pressure and saturation is achieved by

$$p^{n+1} = p^n + (p^n - p^{n-1}) \frac{\Delta t_{new}}{\Delta t_{old}} , \qquad (V\text{-}1)$$

and

$$S^{n+1} = S^n + (S^n - S^{n-1}) \frac{\Delta t_{new}}{\Delta t_{old}} . \qquad (V\text{-}2)$$

To make the solutions so calculated compatible with the implicit finite difference solutions, equation (V-1) and equation (V-2) require a limitation on the time step size selected. Within the range of practical time step sizes, the maximum deviation between predicted and calculated pressures is approximately four psi. For the saturation calculation, the maximum deviation between predicted and calculated values is approximately two percent.

The relative permeabilities are evaluated at an upstream position. A flow check is made before each evaluation, so that there will be no reversal flow calculation; i.e., a phase will not be permitted to flow from a grid element in which that phase saturation is at its irreducible value. This precaution will keep the fluid flux in its proper direction, thereby avoiding material balance error.

When pressure is still above bubble-point pressure, all the terms which constitute the overall compressibility expression (TRM) have the same positive sign. However, when the pressure falls below the bubble point pressure, $(-\frac{S_o}{B_o}\frac{dB_o}{dP})$ will become negative. This should be compensated for by the assumption of a positive, non-zero value of the term $(\frac{S_o B_g}{B_o}\frac{dR_s}{dP})$ which also appears in TRM. However, inaccuracies in implicit coefficient evaluations cause a time lag in effectiveness of this compensation. Therefore, this may cause the value of TRM to become zero or negative, which in turn will prevent the magnitude of the principal diagonal elements of the coefficient matrix from being dominant. To avoid this problem, as the pressure approaches the bubble-point pressure, the sign of TRM is checked. If it is found to be zero or negative, the parameter

$$\left(\frac{S_o}{B_o}\frac{dB_o}{dP}\right)\frac{\partial P}{\partial t}$$

is evaluated as

$$\frac{S_o}{B_o}\frac{\partial B_o}{\partial t},$$

or

$$\left(\frac{S_o}{B_o}\right)^{n+1}_{i,j}\left(B_{o}{}^{n+1}_{i,j} - B_{o}{}^{n}_{i,j}\right)\frac{1}{\Delta t}.$$

The above expression is then added to the right hand side of equation (C-1) and the term $\left(-\frac{S_o}{B_o}\frac{dB_o}{dP}\right)$ is removed from TRM. This procedure is repeated for each time step for which the correction is needed, in order to maintain strict diagonal dominance in the coefficent matrix.

The error in the computed change of volume of oil in place, frequently called material balance error, can be calculated from either the produced volume or the initial in place volume. The former one will be called the material balance error of oil production type (M-B Error of PROD type), and the latter one will be called the material balance of oil-in-place type (M-B Error of OIP type).

$$\text{M-B Error of PROD} = \frac{\sum_j\sum_i (PV^{n}_{i,j}S_{o}{}^{n}_{i,j})/B_{o}{}^{n}_{i,j} - \sum_j\sum_i (PV^{n+1}_{i,j}S_{o}{}^{n+1}_{i,j})/B_{o}{}^{n+1}_{i,j}}{\sum_j\sum_i Q_{o}{}_{i,j}\,\Delta t} - 1, \tag{V-3}$$

$$\text{M-B Error of OIP} = \frac{\sum_j\sum_i (PV^{n+1}_{i,j}S_{o}{}^{n+1}_{i,j})/B_{o}{}^{n+1}_{i,j} + \sum_j\sum_i Q_{o}{}_{i,j}\,\Delta t}{\sum_j\sum_i (PV^{n}_{i,j}S_{o}{}^{n}_{i,j})/B_{o}{}^{n}_{i,j}} - 1. \tag{V-4}$$

If the material balance is correctly maintained, the expression (V-3) must be zero, and (V-4) must also be zero.

The L2-Norm, which is defined as

$$\text{L2-Norm} = \sqrt{\sum_{j}\sum_{i} \text{RESID}^2_{i,j}}/MN, \qquad \text{(V-5)}$$

will give lower values than the normalized residual defined by the equation (IV-5). For the values of maximum residual in the range of 10^{-1} to 10^{-3} atm, L2-Norm values are in the range of 10^{-2} to 10^{-5} atm. However, for the purpose of this study the standardized form of residuals in their L2-Norm was considered unnecessary.

The closure criteria used to complete the iteration for each time step calculation are the following: the maximum normalized absolute value of the residual must be less than a specified tolerance; second, the sum of saturations of the three phases must not differ from unity by more than specified tolerance; and third, the material balance error must be within specified tolerance.

The mathematical model discussed above was used to simulate the field performance of the sample reservoir problem. A hypothetical homogeneous reservoir was considered as the sample problem. This reservoir was one square mile in area and it was divided into 100 blocks of 528 ft x 528 ft size. The reservoir was a structural anticline reservoir. The anticline axis dipped with a slope of 100 ft/mile. The average slopes of the flanks were 40 ft to 60 ft/mile.

The net pay thickness was 20 ft and the permeability was 50 md. The average porosity of the reservoir was 20 percent. The initial oil in place was 8.05×10^6 STB , the initial water in place was 8.30×10^6 STB , and there was no initial gas cap. The pore volume was 19.56×10^6 barrels. In the oil zone the oil saturation was 82

percent of pore volume, and the water saturation was 18 percent, which was the irreducible water saturation. The critical gas saturation was set at 5 percent pore volume, and the residual oil saturation was 18 percent.

Two wells were located at the grid points (3,3) and (3,8), each producing 5,999 barrels of stock tank oil monthly.

The initial reservoir pressure at -6130 ft subsea was 2855 psia, and the bubble-point pressure was 2172 psia. The water compressibility was 3×10^{-6} vol/vol/psi, the rock compressibility was 4×10^{-6} vol/vol/psi, and the undersaturated oil compressibility was 1.5×10^{-5} vol/vol/psi. The oil viscosity at the initial reservoir conditions was 1.08 cp, and initial solution gas-oil ratio was 573 SCF/STB.

The relative permeability and other fluid properties were calculated by the following empirical formulas.

Viscosities (μ) in cp, pressure (P) in psia.

$$\mu_w = .6246, \text{ all pressures,}$$

$$\mu_g = .009 + 5 \times 10^{-6} P,$$

$$\mu_o = 2.239 - 7.16 \times 10^{-4} P - 7.7 \times 10^{-8} P^2, \; P \leq P_{BP},$$

$$\mu_o = 2.239 - 7.16 \times 10^{-4} P_{BP} - 7.7 \times 10^{-8} P_{BP}^2 + 5 \times 10^{-5}(P - P_{BP}),$$

$$P > P_{BP}.$$

Densities (ρ) in lb/cu ft, pressure (P) in psia.

$$\rho_w = 62.275 + 1.875 \times 10^{-4} P,$$

$$\rho_g = 2.932 \times 10^{-3} P,$$

$$\rho_o = 53.03 - 4.87 \times 10^{-3} P - 6.3 \times 10^{-7} P^2, \quad P \leq P_{BP},$$

$$\rho_o = (53.03 - 4.87 \times 10^{-3} P_{BP} - 6.3 \times 10^{-7} P_{BP}^2) \times (1. + C_o \times (P - P_{BP})), \quad P > P_{BP}.$$

Formation volume factors in vol/vol, pressure in psia

$$B_w = 1.002 - 3 \times 10^{-6} P,$$

$$B_g = \frac{14.7}{P},$$

$$B_o = 1.03 + 7.1 \times 10^{-5} P + 5.7 \times 10^{-8} P^2, \quad P \leq P_{BP},$$

$$B_o = (1.03 + 7.1 \times 10^{-5} P_{BP} + 5.7 \times 10^{-8} P_{BP}^2) \times e^{(-C_o (P - P_{BP}))}, \quad P > P_{BP}.$$

Solution gas in SCF/STB, pressure in psia.

$$R_s = 24 + 2.53 \times 10^{-1} P.$$

Relative permeabilities as a fraction, saturation as fraction.

$$k_{rw} = (S_w - S_{wc})/(1. - S_{wc}),$$

$$k_{rg} = \frac{S_g^3 (2. - S_g - 2. S_{wc})}{(1. - S_{wc})^4} \times \frac{(S_g - S_{gc})}{(1. - S_{gc})}, \quad S_w > S_{wc},$$

$$k_{rg} = \frac{S_g^3 (2. - S_g - 2. S_w)}{(1. - S_w)^4} \times \frac{(S_g - S_{gc})}{(1. - S_{gc})}, \quad S_w = S_{wc},$$

$$k_{ro} = \frac{S_o^3 (1. - S_w - 2. S_{wc})}{(1. - S_{wc})^4} \times \frac{(S_o - S_{or})}{(1. - S_{gc})}, \quad S_w > S_{wc},$$

$$k_{ro} = \frac{S_o^4}{(1. - S_{wc})^4} \times \frac{(S_o - S_{or})}{(1. - S_{gc})} , \; S_w = S_{wc} .$$

VI. DISCUSSION OF RESULTS

The simulation of the field performance of the sample reservoir problem was primarily made to investigate the computational efficiency of the residual relaxation scheme that was developed. For the purpose of comparative studies, the pressure equation was also solved by the following methods: LSOR, LSOR+2DC, ADIP, ADEP, SIP, and all combinations of these methods which are compatible with the relaxation scheme. A brief description of the methods employed is presented in Appendix D. Two years of production history were simulated, starting with a time step size of seven days, followed by fifteen days, and continued with thirty days. However, when the pressure approached the bubble-point pressure, instabilities in the material balance calculation were observed. In the region of bubble-point pressure, a higher degree of accuracy of the solutions of the material balance calculation can be obtained by reducing the time step size. A further discussion on the instability of the material balance calculation will be given later in this chapter.

The result of primary concern to this investigation was the behavior of the residuals at each iteration level as the pressure equation was solved by the various methods. The residual level of a solution indicates the degree of accuracy of the solution. In general, the magnitude of residuals depends on the time step size, time period of simulation, and accuracy of the initial pressure estimate.

The results tabulated in Tables I, II, and III show that the residual relaxation method (RELAX) reduced the magnitude of the residuals after

TABLE I

MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS

FOR VARIOUS METHODS, 7-DAY TIME STEP

ITERATION	LSOR	LSOR+2DC	RELAX	RELAX +ADEP	RELAX +ADIP	SIP
1	1.7437	1.7437	.4656	.4291	.1271	.1450
2	1.4583	1.4583	.1778	.2048	.0524	.0341
3	1.1686	1.1686	.0597	.1629	.0229	.0073
4	1.1208	1.1208	.0366	.0827	.0127	.0028
5	.9083	.9295*	.0317	.0395	.0101	.0019
6	.8137	1.0665	.0209	.0185	.0071	.0013
7	.7503	.7998	.0124	.0142	.0053	.0008
8	.6276	.6154	.0069	.0134	.0043	.0004
9	.5185	.6318	.0049	.0126	.0037	.0003
10	.5465	.6010*	.0045	.0118	.0033	.0002
15	.4634	.3875*	.0030	.0082	.0021	.0002
20	.3706	.3327*	.0020	.0055	.0014	.0001
30	.2499	.1544*	.0009	.0024	.0006	.0001
40	.0977	.0816*	.0004	.0011	.0002	
50	.0562	.0500*	.0002	.0005	.0001	
60	.0242	.0272*	.0001	.0002		
70	.0139	.0128*	.0001	.0001		
80	.0058	.0085*				
90	.0039	.0044*				
100	.0019	.0029*				

Note: *) 2DC was applied after every 5 iterations

TABLE II

MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS

FOR VARIOUS METHODS, 15-DAY TIME STEP

ITERATION	LSOR	RELAX +ADEP	RELAX	RELAX +ADIP	SIP
1	3.3220	.4511	.9281	.0651	.3278
2	2.8140	.3553	.3589	.0272	.1038
3	2.2644	.2628	.2006	.0189	.0197
4	2.1407	.1816	.0696	.0119	.0149
5	2.0505	.1250	.0625	.0110	.0116
6	1.7517	.0870	.0553	.0095	.0092
7	1.7467	.0615	.0417	.0073	.0074
8	1.4145	.0444	.0285	.0056	.0058
9	1.1918	.0325	.0117	.0041	.0046
10	1.2631	.0239	.0130	.0030	.0036
15	1.0753	.0060	.0104	.0013	.0011
20	.9650	.0034	.0084	.0010	.0003
30	.3120	.0021	.0053	.0007	.0002
40	.2093	.0013	.0034	.0004	.0001
50	.1112	.0008	.0021	.0003	.0001
60	.0419	.0005	.0013	.0002	
70	.0342	.0003	.0008	.0001	
80	.0164	.0002	.0005		
90	.0150	.0001	.0003		
100	.0137		.0001		

TABLE III

MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS

FOR VARIOUS METHODS, 30-DAY TIME STEP

ITERATION	LSOR	RELAX +ADEP	RELAX	RELAX +ADIP	SIP
1	.5845	.5858	.3689	.1439	.1516
2	.5052	.4960	.1429	.0649	.0436
3	.3871	.3744	.0670	.0372	.0184
4	.4579	.2901	.0523	.0261	.0143
5	.4191	.2293	.0514	.0235	.0119
6	.2382	.1845	.0505	.0198	.0100
7	.3005	.1500	.0480	.0151	.0085
8	.3437	.1224	.0407	.0111	.0073
9	.3478	.1013	.0339	.0079	.0062
10	.2898	.0851	.0281	.0059	.0054
15	.2477	.0400	.0132	.0019	.0025
20	.1652	.0206	.0105	.0011	.0012
30	.0613	.0107	.0077	.0008	.0003
40	.0363	.0082	.0057	.0006	.0002
50	.0174	.0062	.0043	.0005	.0001
60	.0125	.0048	.0032	.0004	.0001
70	.0123	.0034	.0024	.0003	
80	.0091	.0026	.0018	.0002	
90	.0089	.0019	.0013	.0001	
100	.0122	.0014	.0010		

each iteration. These results indicate that the relaxation scheme computed pressures that were closer to the true solution of the pressure difference equation after each iteration. RELAX was able to reduce the maximum residual to the level of 10^{-4} atm without any difficulty. These results demonstrate the feasibility of using RELAX as a numerical method for solving pressure equations.

Tables I, II, and III, and Figure 1 and 2, were constructed for the purpose of comparing the rate of convergence of RELAX with the convergence rate of the other methods. RELAX reduced the maximum residual to 10^{-3} of its initial value within the first 10 iterations. Within the next 10 to 20 iterations the convergence rate of RELAX tapered off. In comparison to LSOR and to LSOR+2DC the maximum residual calculated by RELAX was 10^{-2} times smaller than that calculated by LSOR and LSOR+2DC. In comparison to SIP the residual calculated by RELAX was 10 times bigger than that calculated by SIP. As far as the rate of convergence is concerned, these results indicated that RELAX was much faster than LSOR, and RELAX was somewhat slower than SIP. The closeness of the rate of convergence of RELAX to the rate of convergence of SIP was significant. In order to make a complete comparison between RELAX and SIP, the computer time requirements for each iteration and the number of iterations needed for the maximum residual to converge to a certain level of accuracy were recorded. This time requirement was investigated, since SIP involves calculations which are considerably more complex and therefore require more computational time. These results are tabulated in Table IV. The average time requirement for each iteration (averaged from 300

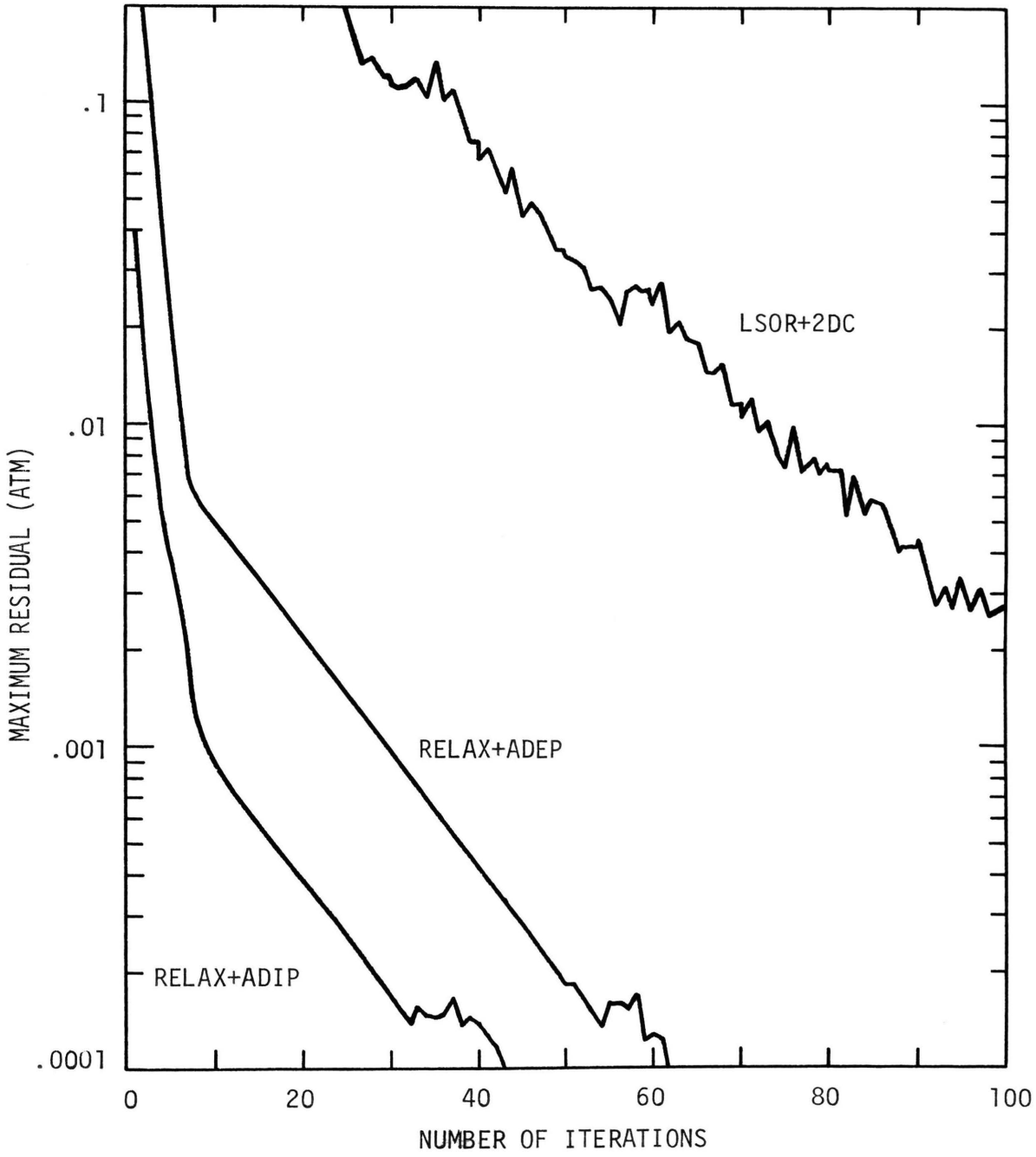

Figure 1. MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS, FOR VARIOUS METHODS, 7-DAY TIME STEP

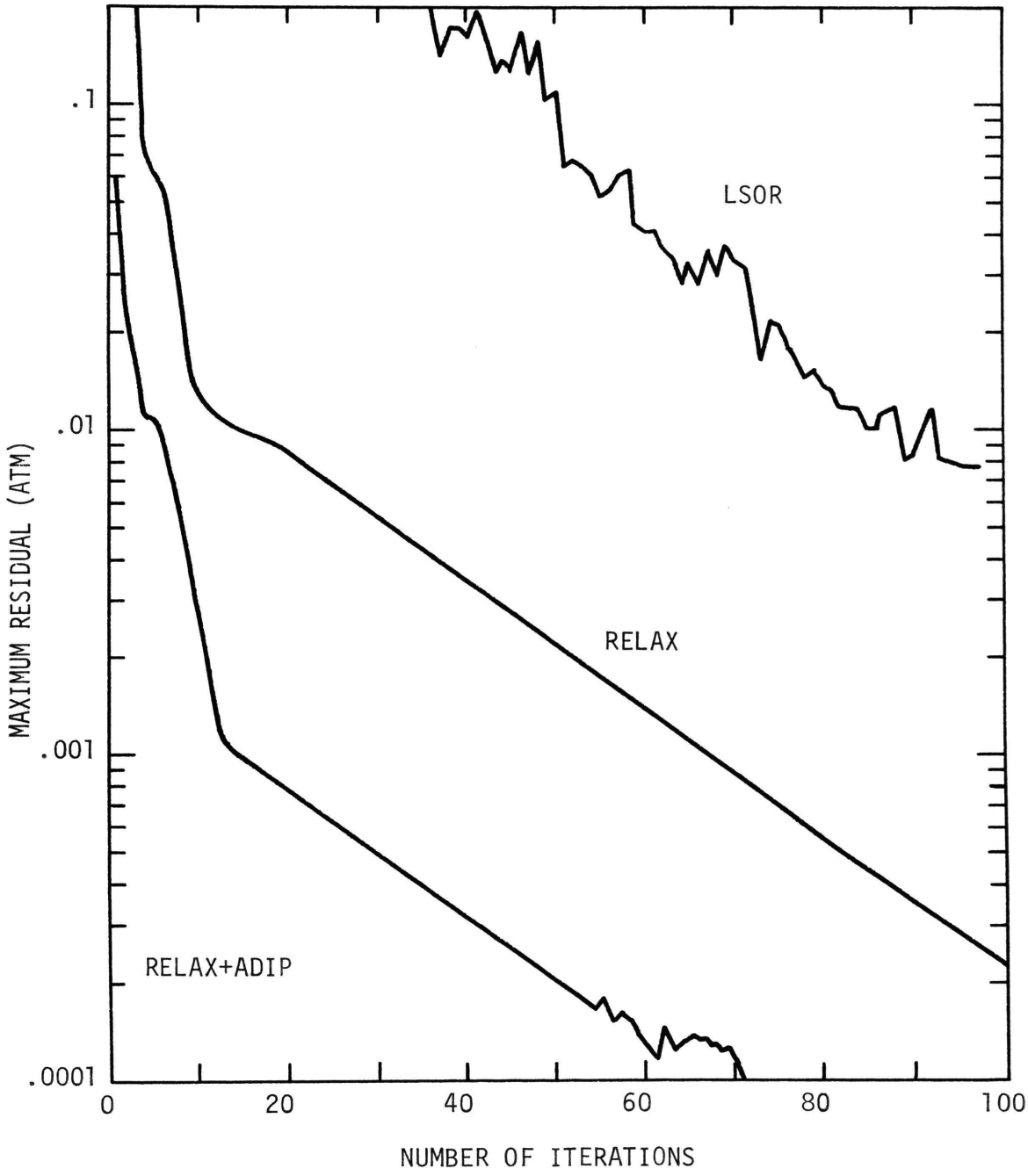

Figure 2. MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS, FOR VARIOUS METHODS, 15-DAY TIME STEP

TABLE IV

COMPUTER TIME REQUIREMENTS (SECONDS)
AND NUMBER OF ITERATIONS NEEDED TO
CONVERGE TO A DESIRED ACCURACY

QUANTITY	RELAX	RELAX+ADIP	SIP	LSOR
Compilation time (seconds)	202.09	211.91	210.86	191.40
Total time (seconds) for 300 iterations	96.19	149.02	279.07	143.58
Average time (seconds) for each iteration	.32	.49	.93	.48
Number of iterations required to converge to 10^{-3} atm maximum residual for:				
7-day time step	25	9	7	>100
15-day time step	54	15	16	>100
30-day time step	77	21	21	>100
Number of iterations required to converge to 10^{-4} atm maximum residual for:				
7-day time step	70	50	30	>100
15-day time step	100	80	50	>100
30-day time step	>100	90	60	>100

Note: Computer time refers to use of IBM 360/50 computer.

iterations) using RELAX was one third of the average time required when SIP was employed. The number of iterations required to reduce the maximum residual to 10^{-3} atm was three times as great for RELAX as for SIP. In order for the maximum residual to be at 10^{-4} atm accuracy level, the number of iterations required in the case of RELAX was more than twice that of SIP. These observations indicated that even though RELAX required more iterations for convergence than did SIP, the amount of total computer time used was approximately the same for the two techniques. However, the large number of iterations required by RELAX is not a favorable feature for a numerical method. Therefore an improvement in the rate of convergence of RELAX was sought.

In order to accelerate the convergence of RELAX, investigations were made to determine the feasibility of using pressures computed by other methods to calculate the residuals to be used as the starting values for RELAX. The combination of ADEP with RELAX was tried and the results are tabulated in Tables I, II, and III and graphed in Figure 3. It was observed that RELAX converged more rapidly than the combination of ADEP+RELAX. Two reasons can be brought forward to explain the failure of the ADEP+RELAX combination: first, ADEP and RELAX are not compatible in the sense of the type of solution technique, since ADEP solves the equation explicitly and RELAX solves it implicitly; second, the ADEP solution gave initial residuals that were higher than those of the first iteration of RELAX.

The results obtained from the combination of RELAX and ADIP were more successful. These tests are summarized by data presented in

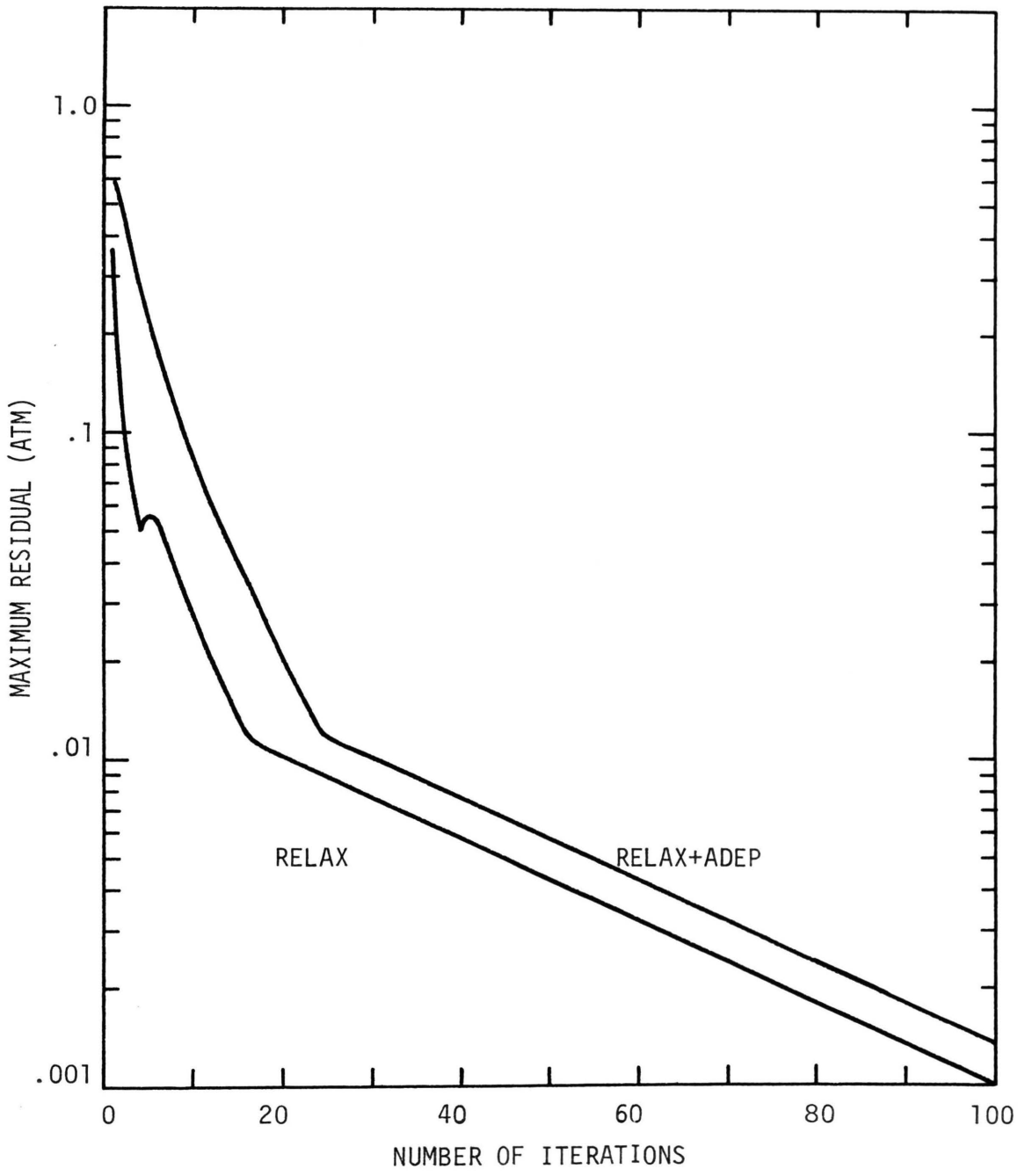

Figure 3. MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS, FOR VARIOUS METHODS, 30-DAY TIME STEP

Tables I, II, and III, as well as Figures 4, 5, and 6. The maximum residual calculated by RELAX+ADIP combination was 10^{-1} times that calculated by RELAX. This improvement made RELAX+ADIP, in the range of 10^{-3} to 10^{-4} atm residual level, converge as rapidly as SIP. The compatibility of RELAX and ADIP can be explained from the facts that both methods solved the same type of implicit equation, and that ADIP gives the lowest magnitude of the initial residuals. As is illustrated by Table IV, the time required for each iteration of RELAX+ADIP was approximately half as much as that needed for SIP. The number of iteration needed to reduce the maximum residual to 10^{-3} atm using RELAX +ADIP was nearly the same as the number needed using SIP. Therefore, it is concluded that RELAX+ADIP is the most efficient technique studied for this accuracy.

To reduce the maximum residual to 10^{-4} atm, the RELAX+ADIP technique required nearly twice as many iterations as SIP. Considering that the time required for each iteration of RELAX+ADIP was half that of SIP, the total computer time requirements of RELAX+ADIP were only slightly less than those of SIP to converge to the pressure solution of the same maximum residual accuracy. Figures 4, 5, and 6 indicate that both methods, RELAX+ADIP and SIP, showed a fluctuation in the magnitudes of the residuals. Considering that the fluctuations occurred at a high iteration, the author believes that the fluctuations were due to the round-off error. It is observed that SIP developed fluctuations of residuals at an earlier iterative level than RELAX +ADIP. The instability phenomenon mentioned above indicated that SIP is more susceptible to round-off error than RELAX+ADIP. This was not

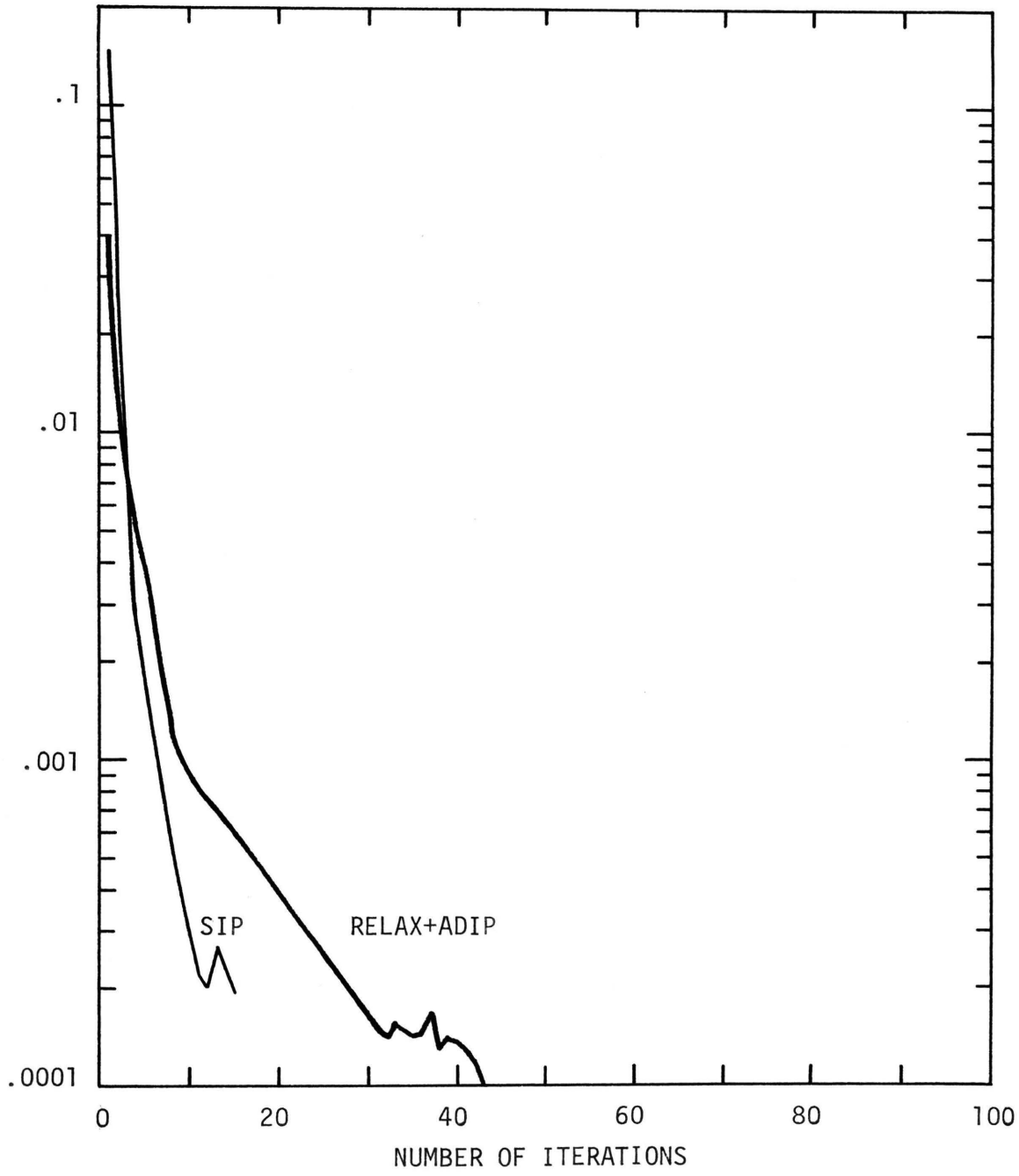

Figure 4. MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS, FOR SIP AND RELAX+ADIP, 7-DAY TIME STEP

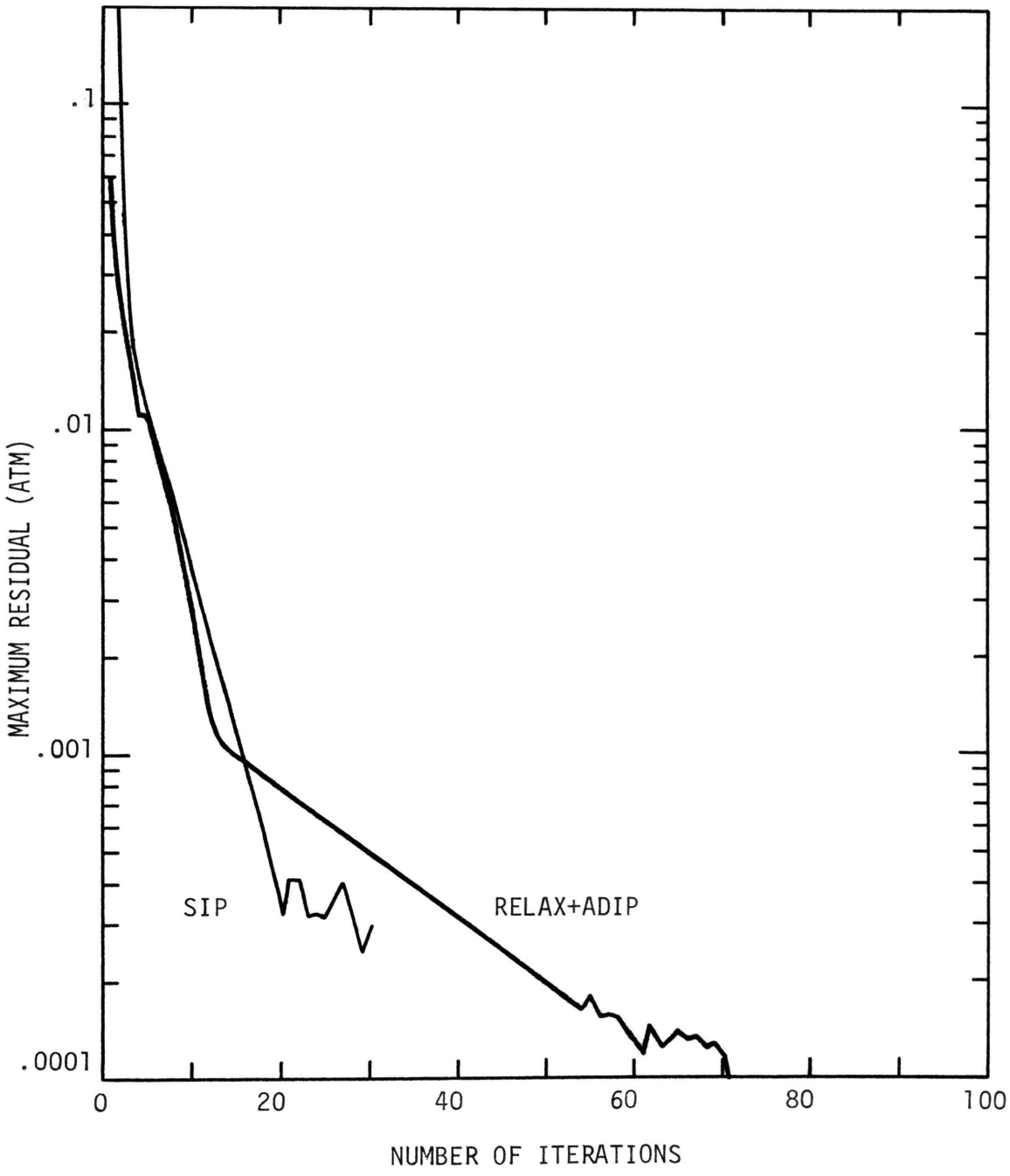

Figure 5. MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS FOR SIP AND RELAX+ADIP, 15-DAY TIME STEP

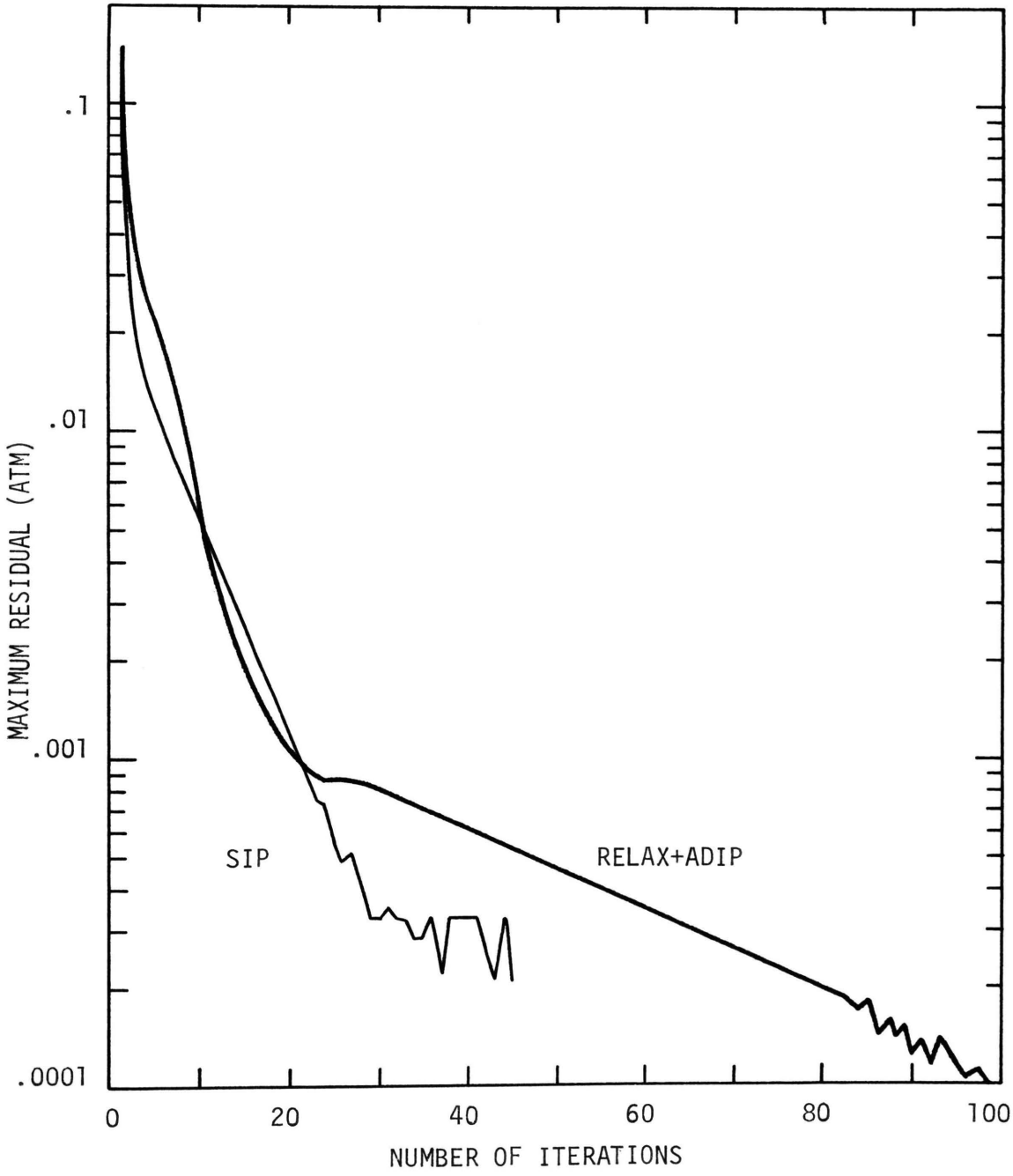

Figure 6. MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS FOR SIP AND RELAX+ADIP, 30-DAY TIME STEP

unexpected, considering the fact that more calculations were performed per iteration in SIP than in RELAX+ADIP.

The author's experience with LSOR was not as successful as the results reported by Watts, and Settari and Aziz. On the other hand, the results of LSOR given here were fully consistent with the results of the investigation made by Stone. In this investigation, LSOR developed fluctuations of residuals which started at an early stage and continued throughout the entire iterations. This situation indicated that LSOR is less stable than the other methods tested, as LSOR was more susceptible to round-off error. The iteration parameter used was calculated by equation (D-7). Trial and error solution was used to arrive at the maximum iteration parameter of 1.93. The need to search for a convergent iteration parameter is another drawback to using LSOR. The 2DC correction method was also applied to LSOR, but did not improve it, as is shown by the results tabulated in Table I.

In using SIP, the iteration parameters were first calculated according to the equation (D-16). By trial and error it was found that values of the iteration parameters used cyclically between 0.0 and .9958 gave a rapid rate of convergence, as is shown in the results of this investigation. The search for finding an optimum iteration parameter added to the computer time needed in using the LSOR and SIP iterative techniques. This additional computational effort was not recorded on the computer time requirements that are presented in Table IV. No iteration parameter is needed when using RELAX+ADIP. This gives it a significant advantage over the other iterative methods.

The accuracy of the RELAX+ADIP technique was affected by the time step size. When the time step was increased, the difference between the initial pressure estimates, obtained by ADIP, and the true solution would be magnified, and the residuals so calculated would be large. The results tabulated in Table V show that the residuals for time step sizes of 45 days or less were about 10^{-1} to 10^{-2} times smaller in magnitude than the residuals for time steps of 60 days or more.

The above discussion has dealt with how each method reduces the residuals after each iteration, and how each method then in turn solves for pressures that are closer to the true solution of the pressure difference equation (C-1). The relationship between the residual level and the material balance accuracy will now be discussed.

The material balance equations (C-2), (C-3), and (C-4) are solved after the pressures have been calculated from equation (C-1). If the material balance is solved using pressures at a high residual level, then the saturations obtained will be in error, since this error will be reflected in the inter-block pressure gradients. Various methods of testing, which are based on material balance principles, were available to determine the accuracy of the saturation calculations. These tests were described in chapter V as the testing methods of the closure criteria.

First consider the obvious material balance test that requires the summation of the saturations in each block to be unity. In deriving equation (C-1) the assumption was made that the summation of the saturation in each block must be one, which in turn caused the saturation derivatives to go to zero. A pressure residual error has the effect of manifesting itself in the saturation calculations as a

TABLE V

MAXIMUM RESIDUAL (ATM) VS. NUMBER OF ITERATIONS
FOR VARIOUS TIME STEP SIZES, BY RELAX+ADIP

ITERATION	TIME STEP SIZES (DAYS)							
	7.5	15	30	45	60	75	90	100
0	.6864	.1809	.2814	.2924	1.1408	2.4464	3.3148	.8607
1	.1314	.0612	.1332	.1262	1.1702	1.0290	.7339	.4168
2	.0566	.0261	.0540	.0609	.9704	.4634	.3882	.3017
3	.0245	.0182	.0313	.0428	.8061	.3840	.2535	.1688
4	.0137	.0113	.0217	.0453	.7363	.3321	.2577	.1238
5	.0111	.0110	.0193	.0482	.6615	.2828	.2246	.0915
10	.0038	.0028	.0058	.0322	.3808	.1617	.0582	.0469
15	.0023	.0009	.0021	.0102	.1362	.1208	.0267	.0397
20	.0015	.0007	.0012	.0034	.0548	.1029	.0236	.0349
30	.0006	.0005	.0009	.0011	.0237	.0839	.0178	.0271
40	.0003	.0003	.0007	.0004	.0115	.0686	.0134	.0209
50	.0001	.0002	.0005	.0002	.0084	.0568	.0100	.0166
60	.0001	.0001	.0003	.0001	.0071	.0471	.0075	.0125
70		.0001	.0002		.0060	.0390	.0056	.0097
80			.0002		.0050	.0323	.0042	.0075
90			.0001		.0042	.0267	.0031	.0058
100			.0001			.0222	.0024	.0045

residual value of the sum of the saturation derivatives. Therefore, low pressure residuals are a necessary condition for satisfying the requirement that saturations sum to unity.

The second test which was employed is based on the principle that the voidage resulting from the decrease in saturation due to production must equal the volume produced. Supposed that there was no error in the calculation of the saturations. Then the total volume of oil remaining in the reservoir plus the oil produced must equal the volume of oil initially in place, where volumes are expressed in terms of standard conditions. This volumetric balance must also be true for the water and the gas phase.

The results that gave the relationship between the residual level and the material balance accuracy are tabulated in Tables VI, VII, and VIII. These data indicate that the material balance accuracy was improved by reducing the residuals. Therefore, it may be concluded that maintenance of an accurate solution to the pressure equation is a necessary condition for an accurate material balance calculation. In Table IX the material balance accuracy is recorded for the entire simulation period, at the maximum residual level of 10^{-3} atm. Table X illustrates the results of material balance accuracy at the maximum residual level of 10^{-4} atm. A comparison of the two tables indicates that no further improvement in the material balance accuracy was observed when the maximum residual level was reduced from 10^{-3} atm to 10^{-4} atm. Tables VI, VII, and VIII, indicate that at the maximum residual level of 10^{-2} atm, the material balance accuracy was still at a low level, lower than if the maximum residual level was 10^{-3} atm.

In the region of bubble-point pressure, a further improvement in

TABLE VI

MATERIAL BALANCE ACCURACY VS. MAXIMUM RESIDUAL

FOR ABOVE AND BELOW BUBBLE POINT CONDITIONS

BY RELAX+ADIP, 7-DAY TIME STEP

ITERATIONS	ABOVE BUBBLE POINT			BELOW BUBBLE POINT		
	MAXIMUM RESIDUAL (ATM)	DEVIATION OF SATURATION SUM FROM 1.00 (%PV)	INCREMENTAL MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)	MAXIMUM RESIDUAL (ATM)	DEVIATION OF SATURATION SUM FROM 1.00 (%PV)	INCREMENTAL MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)
1	.1271	.0147	.001	.6842	1.1114	.0308
2	.0524	.0072	.001	.2860	.6283	.0264
3	.0229	.0041	.001	.1201	.0329	.0258
4	.0127	.0043	.002	.0771	.9017	.0261
5	.0101	.0043	.002	.0681	.8882	.0263
10	.0033	.0041	.002	.0075	.8697	.0264
15	.0021	.0040	.001	.0006	.8686	.0265
20	.0014	.0038	.000	.0001	.8679	.0265
30	.0006	.0037	.000	.0001	.8680	.0260
40	.0002	.0036	.000			
50	.0001	.0036	.000			

TABLE VII

MATERIAL BALANCE ACCURACY VS. MAXIMUM RESIDUAL

FOR ABOVE AND BELOW BUBBLE POINT CONDITIONS

BY RELAX+ADIP, 15-DAY TIME STEP

ITERATIONS	ABOVE BUBBLE POINT			BELOW BUBBLE POINT		
	MAXIMUM RESIDUAL (ATM)	DEVIATION OF SATURATION SUM FROM 1.00 (%PV)	INCREMENTAL MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)	MAXIMUM RESIDUAL (ATM)	DEVIATION OF SATURATION SUM FROM 1.00 (%PV)	INCREMENTAL MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)
1	.0651	.1187	.001	.5620	1.7102	.0111
2	.0272	.1167	.001	.1871	1.0669	.0047
3	.0189	.1163	.001	.1056	.6525	.0025
4	.0119	.1158	.000	.0743	1.3047	.0016
5	.0110	.1153	.000	.0702	.9964	.0007
10	.0030	.1138	.0001	.0211	.9578	.0005
15	.0013	.1132	.0001	.0061	.9472	.0001
20	.0010	.1129	.0001	.0018	.9440	.0003
30	.0007	.1124	.000	.0002	.9440	.0000
40	.0004	.1122	.000	.0001	.9425	.0000
50	.0003	.1120	.0001	.0001	.9425	.0000
60	.0002	.1119	.000			
70	.0001	.1118	.000			

TABLE VIII

MATERIAL BALANCE ACCURACY VS. MAXIMUM RESIDUAL

FOR ABOVE AND BELOW BUBLE POINT CONDITIONS

BY RELAX+ADIP, 30-DAY TIME STEP

ITERATIONS	ABOVE BUBBLE POINT			BELOW BUBBLE POINT		
	MAXIMUM RESIDUAL (ATM)	DEVIATION OF SATURATION SUM FROM 1.00 (%PV)	INCREMENTAL MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)	MAXIMUM RESIDUAL (ATM)	DEVIATION OF SATURATION SUM FROM 1.00 (%PV)	INCREMENTAL MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)
1	.1439	.1117	.001	1.6964	5.1357	2.2723
2	.0649	.0665	.000	1.1982	4.9013	2.0989
3	.0372	.0456	.000	1.1294	4.6250	1.9612
4	.0261	.0664	.000	.9637	4.4901	1.8386
5	.0235	.0655	.001	.8342	4.3372	1.8363
10	.0059	.0650	.000	.3691	3.7432	1.4263
15	.0019	.0649	.0001	.1578	3.5944	1.2864
20	.0011	.0648	.000	.0711	3.4821	1.2214
30	.0008	.0647	.0003	.0156	3.4406	1.1759
40	.0006	.0647	.0002	.0042	3.4066	1.1630
50	.0005	.0647	.000	.0032	3.4007	1.1580
60	.004	.0647	.000	.0024	3.4005	1.1532
70	.0003	.0646	.000	.0018	3.4068	1.1495
80	.0002	.0646	.000	.0013	3.3987	1.1460
90	.0001	.0646	.000	.0010	3.4047	1.1458
100				.0007	3.3977	1.1434

TABLE IX

MATERIAL BALANCE ACCURACY THROUGHOUT THE ENTIRE SIMULATION PERIOD, AT MAXIMUM RESIDUAL OF 10^{-3} ATM, BY RELAX+ADIP

SIMULATION PERIOD (DAYS)	TIME STEP SIZES (DAYS)	ITERATION	DEVIATION OF SATURATION SUM FROM 1.00 (% PV)	INCREMENTAL MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)	CUMULATIVE MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)
7.5	7.5	31	.0040	-.0002	-.0002
22.5	15	19	.1046	.0000	.0112
52.5	30	33	.0652	.0000	.0232
82.5	30	15	.0294	-.0001	.0359
112.5	30	38	1.5119	.0371	.0859
142.5	30	70	2.1264	.1337	.1807
172.5	30	115	2.7122	1.4057	.9237
202.5	30	173	2.0233	1.8126	.8529
232.5	30	101	1.3623	1.3037	-.0568
262.5	30	100	1.2415	1.1585	-.1755
292.5	30	93	1.2342	.9651	-.3428
322.5	30	89	.9614	.8698	-.4189
352.5	30	85	.7928	.7637	-.5017
382.5	30	80	.5870	.6324	-.6145
412.5	30	76	.4101	.5261	-.7073
442.5	30	73	.3459	.4541	-.7682
472.5	30	70	.4510	.4025	-.8106
502.5	30	70	.4413	.3752	-.8261
532.5	30	65	.6271	.3147	-.8712
562.5	30	66	.4999	.2760	-.8882
592.5	30	62	.4693	.1179	-1.0234
622.5	30	61	.3458	.1633	-.9535
652.5	30	62	.2980	.1264	-.9771
682.5	30	62	.2899	.1026	-.9891

TABLE X

MATERIAL BALANCE ACCURACY THROUGHOUT THE ENTIRE SIMULATION PERIOD

AT MAXIMUM RESIDUAL OF 10^{-4} ATM, BY RELAX+ADIP

SIMULATION PERIOD (DAYS)	TIME STEP SIZES (DAYS)	ITERATION	DEVIATION OF SATURATION SUM FROM 1.00 (% PV)	INCREMENTAL MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)	CUMULATIVE MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)
7.5	7.5	60	.0039	.0000	.0000
22.5	15	69	.1039	.0000	.0112
52.5	30	118	.0648	-.0001	.0231
82.5	30	95	.0287	-.0002	.0358
112.5	30	87	1.5119	.0379	.0866
142.5	30	142	2.1251	.1352	.1822
172.5	30	191	2.7111	1.4053	.9240
202.5	30	243	2.9212	1.8091	.8501
232.5	30	145	1.3615	1.3042	-.0594
262.5	30	151	1.2407	1.1591	-.1779
292.5	30	139	1.2331	.9660	-.3452
322.5	30	130	.9614	.8700	-.4210
352.5	30	124	.7922	.7648	-.5040
382.5	30	116	.5865	.6334	-.6170
412.5	30	108	.4100	.5270	-.7091
442.5	30	108	.3460	.4555	-.7699
472.5	30	107	.4512	.4039	-.8125
502.5	30	101	.4412	.3766	-.8280
532.5	30	90	.6203	.3218	-.8675
562.5	30	94	.5008	.2747	-.8934
592.5	30	95	.4719	.1264	-1.0191
622.5	30	91	.3473	.1627	-.9588
652.5	30	90	.2688	.1286	-.9792
682.5	30	95	.2916	.1064	-.9897

the solution of the material balance equations can be made by reducing the magnitude of the discretization error. This was done by reducing the time step size. The effect of this reduction on the accuracy of the material balance solutions is shown in Table XI. These results demonstrate that by subdividing the time period of 112.5 days to 142.5 days into time intervals of 0.5, 1.0, and 2.0 days, the material balance solution was made more accurate. The author did not make any further study on the material balance accuracy as it was not within the scope of this investigation.

In accounting for the results of this investigation, the closure criteria were chosen to be the following: maximum residuals must not exceed 10^{-3} atm, the maximum deviation of the summation of the saturations from 1.00 must be 0.5 percent of pore volume or less, and the maximum material balance error must not exceed 0.5 percent of oil initially in place. Employing these criteria, Table XII shows the results of pressure solutions calculated by the two best methods tested. The pressures calculated by RELAX+ADIP were the same as the pressures calculated by SIP. Considering that both methods achieved a low maximum residual level, it is reasonable to conclude that the RELAX+ADIP and the SIP solutions converge to the numerical solution of the pressure equation. The saturations calculated by RELAX+ADIP were exactly the same as those calculated by SIP. Thus the RELAX+ADIP solutions were virtually identical to the SIP solutions for all cases studied. The simplicity, lack of an interation parameter, and computing speed favor the RELAX+ADIP which was developed in this investigation.

TABLE XI

MATERIAL BALANCE ACCURACY FOR
TIME PERIOD OF 112.5 TO 142.5 DAYS
WITH REDUCED TIME STEP OF 1.0 AND 0.5 DAYS
AT MAXIMUM RESIDUAL OF 10^{-4} ATM, BY
RELAX+ADIP

SIMULATION PERIOD	TIME STEP SIZES	ITERATIONS	DEVIATION OF SATURATION SUM FROM 1.00 (% PV)	INCREMENTAL MATERIAL BALANCE ERROR OF OIL IN PLACE TYPE (% INITIAL OIL IN PLACE)
112.5	2	6	.0006	.0000
114.5	2	3	.0050	.0000
115	.5	2	.0000	.0000
115.5	.5	1	.0000	.0000
116	.5	2	.0000	.0000
116.5	.5	2	.0000	.0000
117	.5	2	.0000	.0000
117.5	.5	2	.0000	.0000
118	.5	2	.0000	.0000
118.5	.5	2	.0000	.0000
119	.5	2	.0000	.0000
119.5	.5	2	.0000	.0000
120	.5	1	.0000	.0000
120.5	.5	2	.0000	.0000
121	.5	2	.0000	.0000
121.5	.5	2	.0000	.0000
122	.5	2	.0000	.0000
122.5	.5	2	.0000	.0000
123.5	1	3	.0000	.0000
124.5	1	7	.9215	.0682
125.5	1	6	.3944	.0141
126.5	1	7	.3948	.0140
127.5	1	7	.2541	.0089
128.5	1	7	.1792	.0063
129.5	1	8	.1237	.0044
130.5	1	10	.0829	.0029
131.5	1	8	.0520	.0018
132.5	1	9	.0278	.0009
134.5	2	17	.4031	.0140
136.5	2	17	.2861	.0100
138.5	2	19	.5508	.0448
140.5	2	17	.3080	.0121
142.5	2	17	.6483	.0936

TABLE XII

PRESSURE HISTORY VS. METHODS, WITH CLOSURE CRITERIA
OF: MAXIMUM RESIDUAL < 1. x 10^{-3} ATM,(SATURATION SUM -1.00) < 0.5 %, AND
M-B ERROR OIL IN PLACE < 0.5 %

TIME PERIOD DAYS	PRESSURES CALCULATED BY RELAX+ADIP(PSI)			PRESSURES CALCULATED BY SIP (PSI)		
	AVERAGE	WELL 3,3	WELL 3,8	AVERAGE	WELL 3,3	WELL 3,8
0	2677.53	2490	2490	2677.53	2490	2490
7	2657.90	2428	2428	2658.01	2428	2428
22	2616.50	2371	2371	2616.46	2371	2371
52	2533.08	2281	2281	2532.74	2281	2281
82	2449.34	2195	2196	2448.89	2195	2195
89	2429.75	2175	2176	2429.31	2175	2176
96	2405.17	2110	2111	2404.75	2110	2111
103	2380.72	2076	2077	2380.31	2076	2076
110	2356.23	2046	2047	2355.81	2046	2047
117	2331.73	2019	2020	2331.31	2019	2020
132	2269.94	1961	1962	2269.52	1960	1962
162	2096.83	1786	1787	2096.92	1786	1788
207	1878.81	1599	1600	1882.88	1601	1602

VII. CONCLUSIONS

A numerical technique has been developed for solving the pressure equation which is employed in numerical simulation of hydrocarbon reservoirs. This iterative technique is based on a method that sequentially reduces the pressure residual at each grid point to zero, and then adjusts the adjacent residuals accordingly. It has been shown analytically that this residual relaxation method (RELAX) is convergent, stable, and consistent with the partial differential equations which describe fluid flow in the reservoir.

A three-phase areal model was developed and was used to simulate performance of a hypothetical homogeneous petroleum reservoir. Comparative studies indicated that:

1. RELAX is efficient, and it will converge to the correct solution of the pressure equation.

2. The total computer time required by RELAX in solving for pressures with an appropriate accuracy is approximately the same as that required by SIP, and is much shorter than is required by LSOR.

3. The RELAX+ADIP combination converges more rapidly than RELAX. The total time required by the RELAX+ADIP solution to converge to a specified residual level is approximately one-half that required by SIP.

4. RELAX+ADIP is stable. The stability of RELAX+ADIP is comparable to that of SIP.

5. RELAX+ADIP is the best method tested for calculating pressures in the simulation model.

BIBLIOGRAPHY

1. Peaceman, D. W., and H. H. Rachford Jr., The Numerical Solution of Parabolic and Elliptic Differential Equations. J. Soc. Ind. Appl. Math., Vol. 3, March 1955, 28.

2. Quon, D., P. M. Dranchuk, S. R. Allada, and P. K. Kleung, Stable, Explicit, Computationally Efficient Method for Solving Two-Dimensional Mathematical Models of Petroleum Reservoirs. Canadian Pet. Tech. J., April-June 1965, 53-58.

3. Coats, K. H., and M. H. Terhune, Comparison of Alternating Direction Explicit and Implicit Procedures in Two-Dimensional Flow Calculations. Soc. Pet. Eng. J., December 1966, 350-362.

4. Bjordammen, J., and K. H. Coats, Comparison of Alternating Direction and Overrelaxation Techniques in Simulation of Reservoir Fluid Flow. Soc. Pet. Eng. J., March 1969, 47-58.

5. Briggs, J. E., and T. N. Dixon, Some Practical Considerations in Numerical Solution of Two-Dimensional Reservoir Problem. Soc. Pet. Eng. J, June 1968, 185-195.

6. Stone, H. L.,Iterative Solution of Implicit Approximations of Multidimensional Partial Differential Equations. SIAM J. on Numerical Analysis, vol. 5, 1968, 530-559.

7. Weinstein, H. G., H. L. Stone, and T. V. Kwan, Simultaneous Solution of Multiphase Reservoir Equations. Soc. Pet. Eng. J., June, 1969, 99-110.

8. Steen, H. D., and S.M.F. Ali, Comparative Evaluation of Strongly Implicit Procedure (SIP) for Gas Reservoir Simulation. Paper SPE 3562 presented at the 46th Annual Fall Meeting of SPE, AIME, New Orleans, La., Oct. 3-6, 1971.

9. Traylor, C., and M. Sheffield, The Comparable Efficiency of Mathematical Techniques Used in Reservoir Simulation. Paper SPE 3560 presented at the 46th Annual Fall Meeting of SPE,AIME, New Orleans, La., Oct. 3-6, 1971.

10. Watts, J. W., A Method for Improving Line Successive Overrelaxation in Anisotropic Problem - A Theoretical Analysis. Paper SPE 3559, presented at 46th Annual Fall Meeting of SPE, AIME, New Orleans, La., Oct. 3-6, 1971.

11. Aziz, K., and A.Settari, A New Iterative Method for Solving Reservoir Simulation Equations. Canada Pet. Eng. J., Jan-March, 1972, 62-68.

12. Routt, K. R., and P. B. Crawford, A New and Fast Method for Solving Large Numbers of Reservoir Simulation Equations. Paper SPE 4277, presented at the 3rd Numerical Simulation Symposium of SPE, AIME, Houston, Tex., Jan. 10-12, 1973.

13. Southwell, R. V., Stress Calculation in Frameworks by the Method of Systematic Relaxation of Constraints, I and II, Proc. Roy. Soc. A, vol. 153, 1935, 41-76.

14. Temple, G., The General Theory of Relaxation Methods Applied to Linear Systems. Proc. Roy. Soc. A, vol. 169, 1939, 476-500.

15. Emmons, H. W., The Numerical Solution of Heat Conduction Problems. Trans. ASME, 1943, 607-612.

16. Dykstra, H., and R. L. Parsons, Relaxation Method Applied to Oil Field Research. Trans. AIME, 1951, 227-232.

17. Mitchell, A. R., and D. E. Rutherford, On the Theory of Relaxation. Proc. Glasgow Math. Ass., 1953, 101-110.

18. Allen, D. N. deG., Relaxation Methods. London: McGraw Hill Book Company, Inc., 1954.

19. Scarborough, J. B., Numerical Mathematical Analysis. Baltimore: The Johns Hopkins Press, 1955, 328-338.

20. Lowan, A. N., The Operator Approach to Problems of Stability and Convergence. New York: Scripta Mathematica, 1957, 47-48.

21. Smith, G. D., Numerical Solution of Partial Differential Equations. London: Oxford University Press, 1965, 55-89.

VITA

Grufron Achmad was born on April 11, 1935, in Pangkalpinang, Indonesia. He received his college education from the Institute of Technology Bandung, in Bandung, Indonesia; the University of Oklahoma in Norman, Oklahoma; and the University of Missouri in Rolla, Missouri. He received a Bachelor of Science degree in Mining Engineering from the Institute of Technology Bandung in August 1959; and a Master of Science degree in Petroleum Engineering from the University of Missouri-Rolla in May 1970.

His professional experiences have been with the Institute of Technology Bandung as Secretary, Mining Engineering Department for the period 1960-1962; Head, Petroleum Engineering Department, for the period 1962-1968; Deputy Dean, School of Mineral Technology, for the period 1962-1969; with the National Institute of Geology and Mining in Bandung, Indonesia, as Assistant Director for the period 1963-1969; and with the Indonesian Petroleum Institute in Jakarta, Indonesia, as a member of the Advisory Board, for the period 1965-1969.

APPENDICES

APPENDIX A

DERIVATION OF PRESSURE EQUATION

The mass conservation equation for each elemental block of the reservoir may be stated as follows:

(Mass rate in) - (Mass rate out) + (Mass rate of production)
= (Mass rate of accumulation)

Using Darcy's law (in cgs units) to describe the flow characteristics of the above mass balance yields a mass rate equation for an areal model in the positive X-direction as follows:

$$q_x\rho = - \frac{k_x A\rho}{\mu B} \frac{\partial \Phi}{\partial x} ,$$

where x is the linear distance in the direction of flow, q_x is the volumetric flow rate in the X-direction evaluated at standard conditions, and ρ is the fluid density at standard conditions. The mass rate equation in the positive Y-direction is the same as that in the X-direction, except for the subscript; i.e.,

$$q_y\rho = - \frac{k_y A\rho}{\mu B} \frac{\partial \Phi}{\partial y} .$$

For a two-dimensional areal model, the total rate of mass influx is the sum of the influxes in both directions, i.e.,

$$\text{Mass rate out} = (q_x\rho + q_y\rho)_{OUT}$$

Mass rate out can also be written as:

$$\text{Mass rate out} = \text{Mass rate in} + \Delta x \frac{\partial}{\partial x} (q_x\rho) - \Delta y \frac{\partial}{\partial y} (q_y\rho).$$

Mass production rate = - $Q\rho$, where Q is the well production rate at standard conditions. In the case of an injection well the sign is positive.

$$\text{Mass accumulation rate} = \frac{\partial}{\partial t}\left\{\frac{(PV)\ S\rho}{B}\right\} \quad ,$$

where pore volume (PV) = ($\Delta x \Delta y\ h\phi$), ϕ is porosity expressed as a fraction of the total volume, and S is fluid saturation expressed as a fraction of pore volume.

Considering three fluid phases flowing, the conservation equation for each phase can be written in the following manner:

$$-B\Delta x \frac{\partial}{\partial x}(q_x\rho) - B\Delta y \frac{\partial}{\partial y}(q_y\rho) - QB\rho = B \frac{\partial}{\partial t}\left(\frac{\Delta x \Delta y\ h\phi}{B}\ S\rho\right).$$

After division by ρ, which is a constant at standard conditions, the conservation equation becomes:

$$-B\Delta x \frac{\partial}{\partial x}(q_x) - B\Delta y \frac{\partial}{\partial y}(q_y) - QB = B \frac{\partial}{\partial t}\left(\frac{\Delta x \Delta y\ h\phi}{B}\ S\right)$$

Remembering that $k_{eff} = k_{abs} \times k_r$, the conservation equation for the water phase is as follows:

$$B_w \frac{\partial}{\partial x}\left(\frac{k_x h\ k_{rw}}{\mu_w B_w} \frac{\partial \Phi_w}{\partial x}\right) +$$

$$B_w \frac{\partial}{\partial y}\left(\frac{k_y h\ k_{rw}}{\mu_w B_w} \frac{\partial \Phi_w}{\partial y}\right) - \frac{Q_w B_w}{\Delta x \Delta y} =$$

$$h\phi \left(\frac{S_w}{\phi}\frac{d\phi}{dP} - \frac{S_w}{B_w}\frac{dB_w}{dP}\right)\frac{\partial P}{\partial t} + h\phi \frac{\partial S_w}{\partial t} \ . \qquad \text{(A-1)}$$

The equation for the oil phase is the same as that for the water phase, except for the subscripts.

$$B_o \frac{\partial}{\partial x}\left(\frac{k_x h\, k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial x}\right) +$$

$$B_o \frac{\partial}{\partial y}\left(\frac{k_y h\, k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial y}\right) - \frac{Q_o B_o}{\Delta x \Delta y} =$$

$$h\phi \left(\frac{S_o}{\phi}\frac{d\phi}{dP} - \frac{S_o}{B_o}\frac{dB_o}{dP}\right)\frac{\partial P}{\partial t} + h\phi \frac{\partial S_o}{\partial t} . \qquad \text{(A-2)}$$

In deriving the equation for the gas phase, a total gas balance was made; i.e. both free gas and solution gas were taken into account. The result of this derivation is

$$B_g \frac{\partial}{\partial x}\left(\frac{k_x h\, k_{rg}}{\mu_g B_g}\frac{\partial \Phi_g}{\partial x}\right) + B_g \frac{\partial}{\partial y}\left(\frac{k_y h\, k_{rg}}{\mu_g B_g}\frac{\partial \Phi_g}{\partial y}\right) +$$

$$B_g \frac{\partial}{\partial x}\left(\frac{k_x h\, k_{ro}}{\mu_o B_o} R_s \frac{\partial \Phi_o}{\partial x}\right) + B_g \frac{\partial}{\partial y}\left(\frac{k_y h\, k_{ro}}{\mu_o B_o} R_s \frac{\partial \Phi_o}{\partial y}\right) -$$

$$R_s B_g \frac{\partial}{\partial x}\left(\frac{k_x h\, k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial x}\right) - R_s B_g \frac{\partial}{\partial y}\left(\frac{k_y h\, k_{ro}}{\mu_o B_o}\frac{\partial \Phi_o}{\partial y}\right) -$$

$$\frac{Q_g B_g}{\Delta x \Delta y} = h\phi \left(\frac{S_g}{\phi}\frac{d\phi}{dP} - \frac{S_g}{B_g}\frac{dB_g}{dP} + \frac{S_o}{B_o} B_g \frac{dR_s}{dP}\right)\frac{\partial P}{\partial t}$$

$$+ h\phi \frac{\partial S_g}{\partial t} , \qquad \text{(A-3)}$$

where Q_g is the rate of production of gas which is free gas in the reservoir.

The pressure equation is derived by adding the mass conservation equations of the three phases. Since the sum of saturations must equal unity, it follows that $\frac{\partial}{\partial t}(S_w + S_o + S_g) = 0$. Also note that $\frac{1}{\phi}\frac{d\phi}{dP} = C_{pv}$, and that $\frac{1}{B_w}\frac{dB_w}{dP} = C_w$. Taking into account the effects of gravity and capillary pressure, and defining the dependent variable P as the oil phase pressure and defining Φ in terms of oil phase

pressure ($P \equiv P_o$), then

$$\Phi_o = P - \rho_o gz,$$

$$\Phi_w = P - P_{cow} - \rho_w gz,$$

$$\Phi_g = P + P_{cog} - \rho_g gz,$$

where z is taken to be positive downward. The pressure equation can be rearranged as follows:

$$B_w \frac{\partial}{\partial x}\left(\sigma_{xw} \frac{\partial P}{\partial x}\right) + B_w \frac{\partial}{\partial y}\left(\sigma_{yw} \frac{\partial P}{\partial y}\right) +$$

$$B_o \frac{\partial}{\partial x}\left(\sigma_{xo} \frac{\partial P}{\partial x}\right) + B_o \frac{\partial}{\partial y}\left(\sigma_{yo} \frac{\partial P}{\partial y}\right) +$$

$$B_g \frac{\partial}{\partial x}\left(\sigma_{xg} \frac{\partial P}{\partial x}\right) + B_g \frac{\partial}{\partial y}\left(\sigma_{yg} \frac{\partial P}{\partial y}\right) +$$

$$B_g \frac{\partial}{\partial x}\left(\sigma_{xo} R_s \frac{\partial P}{\partial x}\right) + B_g \frac{\partial}{\partial y}\left(\sigma_{yo} R_s \frac{\partial P}{\partial y}\right) -$$

$$B_g R_s \frac{\partial}{\partial x}\left(\sigma_{xo} \frac{\partial P}{\partial x}\right) - B_g R_s \frac{\partial}{\partial y}\left(\sigma_{yo} \frac{\partial P}{\partial y}\right) =$$

$$(QTERM + PCTERM + GTERM) + TRM \frac{\partial P}{\partial t}, \qquad (A\text{-}4)$$

where

$$\sigma_{xp} = \frac{(k_x h)\, k_{rp}}{\mu_p B_p}, \qquad (A\text{-}5)$$

$$\sigma_{yp} = \frac{(k_y h)\, k_{rp}}{\mu_p B_p}, \qquad (A\text{-}6)$$

$$QTERM = \frac{Q_w B_w + Q_o B_o + Q_g B_g}{\Delta x \Delta y} , \qquad (A-7)$$

$$PCTERM = B_w \frac{\partial}{\partial x} \left(\sigma_{xw} \frac{\partial P_{cow}}{\partial x}\right) + B_w \frac{\partial}{\partial y} \left(\sigma_{yw} \frac{\partial P_{cow}}{\partial y}\right) -$$

$$B_g \frac{\partial}{\partial x} \left(\sigma_{xg} \frac{\partial P_{cog}}{\partial x}\right) - B_g \frac{\partial}{\partial y} \left(\sigma_{yg} \frac{\partial P_{cog}}{\partial y}\right) , \qquad (A-8)$$

$$GTERM = B_w \frac{\partial}{\partial x} \left(\sigma_{xw} \frac{\partial \rho_w gz}{\partial x}\right) + B_w \frac{\partial}{\partial y} \left(\sigma_{yw} \frac{\partial \rho_w gz}{\partial y}\right) +$$

$$B_o \frac{\partial}{\partial x} \left(\sigma_{xo} \frac{\partial \rho_o gz}{\partial x}\right) + B_o \frac{\partial}{\partial y} \left(\sigma_{yo} \frac{\partial \rho_o gz}{\partial y}\right) +$$

$$B_g \frac{\partial}{\partial x} \left(\sigma_{xg} \frac{\partial \rho_g gz}{\partial x}\right) + B_g \frac{\partial}{\partial y} \left(\sigma_{yg} \frac{\partial \rho_g gz}{\partial y}\right) +$$

$$B_g \frac{\partial}{\partial x} \left(\sigma_{xo} R_s \frac{\partial \rho_o gz}{\partial x}\right) + B_g \frac{\partial}{\partial y} \left(\sigma_{yo} R_s \frac{\partial \rho_o gz}{\partial y}\right) -$$

$$B_g R_s \frac{\partial}{\partial x} \left(\sigma_{xo} \frac{\partial \rho_o gz}{\partial x}\right) - B_g R_s \frac{\partial}{\partial y} \left(\sigma_{yo} \frac{\partial \rho_o gz}{\partial y}\right) , \qquad (A-9)$$

$$TRM = C_{pv} + C_w S_w - \frac{S_g}{B_g} \frac{dB_g}{dP} - \frac{S_o}{B_o} \frac{dB_o}{dP} +$$

$$\frac{S_o}{B_o} B_g \frac{dR_s}{dP} , \qquad (A-10)$$

The solution of a system of equations obtained from equation (A-4) written at each point in the grid system yields the values of pressures for each grid location at the new time level.

APPENDIX B

DERIVATION OF MATERIAL BALANCE EQUATION

The material balance equation was derived by writing the mass conservation equation for each phase and solving for the saturations explicitly.

The right-hand side of equation (A-1) can be written as

$$h\phi \frac{\partial S_w}{\partial t} + h\phi S_w (C_{pv} + C_w) \frac{\partial P}{\partial t} .$$

Dividing equation (A-1) by ($h\phi$) yields a material balance equation for the water phase as follows:

$$\frac{\partial S_w}{\partial t} + S_w(C_{pv} + C_w) \frac{\partial P}{\partial t} =$$

$$\frac{1}{\phi h} \{- \frac{Q_w B_w}{\Delta x \Delta y} + B_w \frac{\partial}{\partial x} \sigma_{xw} \frac{\partial}{\partial x} (P - P_{cow} - \rho_w gz) +$$

$$B_w \frac{\partial}{\partial y} \sigma_{yw} \frac{\partial}{\partial y} (P - P_{cow} - \rho_w gz)\}. \qquad \text{(B-1)}$$

The material balance equation for the oil phase was derived in a similar manner. The following result was achieved:

$$\frac{\partial S_o}{\partial t} + S_o(C_{pv} - \frac{1}{B_o} \frac{dB_o}{dP}) \frac{\partial P}{\partial t} =$$

$$\frac{1}{\phi h} \{- \frac{Q_o B_o}{\Delta x \Delta y} + B_o \frac{\partial}{\partial x} \sigma_{xo} \frac{\partial}{\partial x} (P - \rho_o gz) +$$

$$B_o \frac{\partial}{\partial y} \sigma_{yo} \frac{\partial}{\partial y} (P - \rho_o gz)\}. \qquad \text{(B-2)}$$

Similarly, for the gas phase, the material balance is:

$$\frac{\partial S_g}{\partial t} + S_g\left(C_{pv} - \frac{1}{B_g}\frac{dB_g}{dP}\right)\frac{\partial P}{\partial t} + \frac{S_o}{B_o} B_g \frac{dR_s}{dP}\frac{\partial P}{\partial t} =$$

$$\frac{1}{\phi h}\left\{-\frac{Q_g B_g}{\Delta x \Delta y} + B_g \frac{\partial}{\partial x}\sigma_{xg}\frac{\partial}{\partial x}(P + P_{cog} - \rho_o gz) +\right.$$

$$B_g \frac{\partial}{\partial y}\sigma_{yg}\frac{\partial}{\partial y}(P + P_{cog} - \rho_o gz) +$$

$$B_g \frac{\partial}{\partial x}\sigma_{xo} R_s \frac{\partial}{\partial x}(P - \rho_o gz) + B_g \frac{\partial}{\partial y}\sigma_{yo} R_s \frac{\partial}{\partial y}(P - \rho_o gz)$$

$$\left. - B_g R_s \frac{\partial}{\partial x}\sigma_{xo}\frac{\partial}{\partial x}(P - \rho_o gz) - B_g R_s \frac{\partial}{\partial y}\sigma_{yo}\frac{\partial}{\partial y}(P - \rho_o gz)\right\}.$$

(B-3)

APPENDIX C

THE FINITE DIFFERENCE EQUATIONS

The pressure equation and the material balance equations are solved by finite difference approximations. For a rectangular coordinate grid system (i,j), and equal distance increments in the X-direction and Y-direction, the finite difference equation which is used to solve the pressure equation can be expressed as follows:

$$AX_{i,j}\, P_{i-1,j} + (-AX_{i,j} - CX_{i,j})\, P_{i,j} + CX_{i,j}\, P_{i+1,j} +$$

$$AY_{i,j}\, P_{i,j-1} + (-AY_{i,j} - CY_{i,j})\, P_{i,j} + CY_{i,j}\, P_{i,j+1} =$$

$$(QTERM_{i,j} + PCTERM_{i,j} + GTERM_{i,j}) +$$

$$\frac{TRM_{i,j}}{\Delta t}\ (P^{n+1}_{i,j} - P^{n}_{i,j}), \qquad \text{(C-1)}$$

where the terms are as follows:

$$AX_{i,j} = \frac{1}{\Delta x^2}\{B_{w_{i,j}}\, \sigma_{xw_{i-\frac{1}{2},j}} + B_{g_{i,j}}\, \sigma_{xg_{i-\frac{1}{2},j}} +$$

$$(B_{o_{i,j}} + B_{g_{i,j}} R_{s_{i-\frac{1}{2},j}} - B_{g_{i,j}} R_{s_{i,j}})\sigma_{xo_{i-\frac{1}{2},j}}\},$$

$$CX_{i,j} = \frac{1}{\Delta x^2}\{B_{w_{i,j}}\, \sigma_{xw_{i+\frac{1}{2},j}} + B_{g_{i,j}}\, \sigma_{xg_{i+\frac{1}{2},j}} +$$

$$(B_{o_{i,j}} + B_{g_{i,j}} R_{s_{i+\frac{1}{2},j}} - B_{g_{i,j}} R_{s_{i,j}})\, \sigma_{xo_{i+\frac{1}{2},j}}\},$$

$$AY_{i,j} = \frac{1}{\Delta y^2}\{B_{w_{i,j}}\sigma_{yw_{i,j-\frac{1}{2}}} + B_{g_{i,j}}\sigma_{yg_{i,j-\frac{1}{2}}} +$$

$$(B_{o_{i,j}} + B_{g_{i,j}}R_{s_{i,j-\frac{1}{2}}} - B_{g_{i,j}}R_{s_{i,j}})\sigma_{yo_{i,j-\frac{1}{2}}}\},$$

$$CY_{i,j} = \frac{1}{\Delta y^2}\{B_{w_{i,j}}\sigma_{yw_{i,j+\frac{1}{2}}} + B_{g_{i,j}}\sigma_{yg_{i,j+\frac{1}{2}}} +$$

$$(B_{o_{i,j}} + B_{g_{i,j}}R_{s_{i,j+\frac{1}{2}}} - B_{g_{i,j}}R_{s_{i,j}})\sigma_{yo_{i,j+\frac{1}{2}}}\},$$

$$PCTERM_{i,j} = \frac{1}{\Delta x^2}\{B_{w_{i,j}}\sigma_{xw_{i-\frac{1}{2},j}}(P_{cow_{i-1,j}} - P_{cow_{i,j}}) -$$

$$B_{w_{i,j}}\sigma_{xw_{i+\frac{1}{2},j}}(P_{cow_{i,j}} - P_{cow_{i+1,j}}) +$$

$$B_{g_{i,j}}\sigma_{xg_{i-\frac{1}{2},j}}(P_{cog_{i-1,j}} - P_{cog_{i,j}}) -$$

$$B_{g_{i,j}}\sigma_{xg_{i+\frac{1}{2},j}}(P_{cog_{i,j}} - P_{cog_{i+1,j}}) +$$

$$\frac{1}{\Delta y^2}\{B_{w_{i,j}}\sigma_{yw_{i,j-\frac{1}{2}}}(P_{cow_{i,j-1}} - P_{cow_{i,j}}) -$$

$$B_{w_{i,j}}\sigma_{yw_{i,j+\frac{1}{2}}}(P_{cow_{i,j}} - P_{cow_{i,j+1}}) +$$

$$B_g\sigma_{yg_{i,j-\frac{1}{2}}}(P_{cog_{i,j-1}} - P_{cog_{i,j}}) -$$

$$B_g\sigma_{yg_{i,j+\frac{1}{2}}}(P_{cog_{i,j}} - P_{cog_{i,j+1}})\},$$

$$QTERM_{i,j} = \frac{1}{\Delta x \Delta y} \{Q_o B_o +$$

$$\left(\frac{k_{rg}}{\mu_g} + \frac{k_{rw}}{\mu_w}\right) Q_o \left(\frac{\mu_o B_o}{k_{ro}}\right)\},$$

$$GTERM_{i,j} = \frac{1}{\Delta x^2} \{B_{w_{i,j}} \sigma_{xw_{i-\frac{1}{2},j}} (\rho_w g)_{i-\frac{1}{2},j} +$$

$$B_{g_{i,j}} \sigma_{xg_{i-\frac{1}{2},j}} (\rho_g g)_{i-\frac{1}{2},j} +$$

$$(B_{o_{i,j}} + B_{g_{i,j}} R_{s_{i-\frac{1}{2},j}} - B_{g_{i,j}} R_{s_{i,j}}) \sigma_{xo_{i-\frac{1}{2},j}}$$

$$(\rho_o g)_{i-\frac{1}{2},j}\} (z_{i-1,j} - z_{i,j}) - \frac{1}{\Delta x^2}$$

$$\{B_{w_{i,j}} \sigma_{xw_{i+\frac{1}{2},j}} (\rho_w g)_{i+\frac{1}{2},j} +$$

$$B_{g_{i,j}} \sigma_{xg_{i+\frac{1}{2},j}} (\rho_g g)_{i+\frac{1}{2},j} +$$

$$(B_{o_{i,j}} + B_{g_{i,j}} R_{s_{i+\frac{1}{2},j}} - B_{g_{i,j}} R_{s_{i,j}}) \sigma_{xo_{i+\frac{1}{2},j}}$$

$$(\rho_o g)_{i+\frac{1}{2},j}\} (z_{i,j} - z_{i+1,j}) +$$

$$\frac{1}{\Delta y^2} \{B_{w_{i,j}} \sigma_{yw_{i,j-\frac{1}{2}}} (\rho_w g)_{i,j-\frac{1}{2}} +$$

$$B_{g_{i,j}} \sigma_{xg_{i,j-\frac{1}{2}}} (\rho_g g)_{i,j-\frac{1}{2}} +$$

$$(B_{o_{i,j}} + B_{g_{i,j}} R_{s_{i,j-\frac{1}{2}}} - B_{g_{i,j}} R_{s_{i,j}}) \sigma_{yo_{i,j-\frac{1}{2}}}$$

$$(\rho_o g)_{i,j-\frac{1}{2}}\}(z_{i,j-1} - z_{i,j}) - \{B_{w_{i,j}}\ \sigma_{yw_{i,j+\frac{1}{2}}}\ (\rho_w g)_{i,j+\frac{1}{2}} +$$

$$B_{g_{i,j}}\sigma_{yg_{i,j+\frac{1}{2}}}\ (\rho_g g)_{i,j+\frac{1}{2}} + (B_{o_{i,j}} + B_{g_{i,j+\frac{1}{2}}} R_{s_{i,j-\frac{1}{2}}} - B_{g_{i,j}} R_{s_{i,j}})$$

$$\sigma_{yo_{i,j+\frac{1}{2}}}\ (\rho_o g)_{i,j+\frac{1}{2}}\}(z_{i,j} - z_{i,j+1})\ \frac{1}{\Delta y^2}\ ,$$

$$TRM_{i,j} = C_{pv} + C_w\ S_{w_{i,j}} - (\frac{S_g}{B_g})_{i,j}(\frac{dB_g}{dP})_{i,j} +$$

$$(\frac{S_o}{B_o}\ B_g)_{i,j}\ (\frac{dR_s}{dP})_{i,j} - (\frac{S_o}{B_o})_{i,j}\ (\frac{dB_o}{dP})_{i,j}\ .$$

The coefficient of $P_{i,j}^{n+1}$, which is made up of combinations of the other coefficients, will vary according to the method used to obtain a solution. The details of these approaches are documented in the literature with the exception of the method developed in this investigation. The new approach is discussed in Chapter IV.

The finite difference approximations which are employed to solve the material balance equations are as follows:

$$S_{w_{i,j}}^{n+1} = \frac{1}{1 + (C_{pv} + C_w)(P_{i,j}^{n+1} - P_{i,j}^{n})}\ (S_{w_{i,j}}^{n} + \frac{\Delta t}{(\phi h)_{i,j}}$$

$$\{-\ \frac{Q_{w_{i,j}}\ B_{w_{i,j}}}{\Delta x \Delta y} + \frac{B_{w_{i,j}}}{\Delta x^2}(\sigma_{xw_{i-\frac{1}{2},j}}\ \{P_{i-1,j} - P_{i,j}$$

$$- P_{cow_{i-1,j}} + P_{cow_{i,j}} - (\rho_w g)_{i-\frac{1}{2},j}\ (z_{i-1,j} -$$

$$z_{i,j})\} - \sigma_{xw_{i+\frac{1}{2},j}}\{P_{i,j} - P_{i+1,j} - P_{cow_{i,j}} + P_{cow_{i+1,j}} -$$

$$(\rho_w g)_{i+\frac{1}{2},j}\,(z_{i,j} - z_{i+1})\}) + \frac{B_{w_{i,j}}}{\Delta y^2}\,(\sigma_{yw_{i,j-\frac{1}{2}}}\{P_{i,j-1} -$$

$$P_{i,j} - P_{cow_{i,j-1}} + P_{cow_{i,j}} - (\rho_w g)_{i,j-\frac{1}{2}}\,(z_{i,j-1} - z_{i,j})\} -$$

$$\sigma_{yw_{i,j+\frac{1}{2}}}\{P_{i,j} - P_{i,j+1} - P_{cow_{i,j}} + P_{cow_{i,j+1}} - (\rho_w g)_{i,j+\frac{1}{2}}$$

$$(z_{i,j} - z_{i,j+1})\})\}), \qquad \text{(C-2)}$$

$$S_{o_{i,j}}^{n+1} = \frac{1}{1 + \{C_{pv} - \frac{1}{B_{o_{i,j}}}\,(\frac{dB_o}{dP})_{i,j}\}(P_{i,j}^{n+1} - P_{i,j}^{n})}$$

$$(S_{o_{i,j}}^{n} + \frac{\Delta t}{(\phi h)_{i,j}}\{-\frac{Q_{o_{i,j}} B_{o_{i,j}}}{\Delta x \Delta y} + \frac{B_{o_{i,j}}}{\Delta x^2}$$

$$(\sigma_{xo_{i-\frac{1}{2},j}}\{P_{i-1,j} - P_{i,j} - (\rho_o g)_{i-\frac{1}{2},j}\,(z_{i-1,j} - z_{i,j})\}$$

$$- \sigma_{xo_{i+\frac{1}{2},j}}\{P_{i,j} - P_{i+1,j} - (\rho_o g)_{i+\frac{1}{2},j}\,(z_{i,j} - z_{i+1,j})\})$$

$$+ \frac{B_{o_{i,j}}}{\Delta y^2}\,(\sigma_{yo_{i,j-\frac{1}{2}}}\{P_{i,j-1} - P_{i,j} - (\rho_o g)_{i,j-\frac{1}{2}}$$

$$(z_{i,j-1} - z_{i,j})\} - \sigma_{oy_{i,j+\frac{1}{2}}}\{P_{i,j} - P_{i,j+1}$$

$$- (\rho_o g)_{i,j+\frac{1}{2}}\,(z_{i,j} - z_{i,j+1})\})\}), \qquad \text{(C-3)}$$

$$S_{g_{i,j}}^{n+1} = \frac{1}{1 + \{C_{pv} - \frac{1}{B_{g_{i,j}}} (\frac{dB_g}{dP})_{i,j}\}(P_{i,j}^{n+1} - P_{i,j}^{n})}$$

$$(S_{g_{i,j}}^{n} - \frac{S_{o_{i,j}}}{B_{o_{i,j}}} B_{g_{i,j}} (\frac{dR_s}{dP})_{i,j} (P_{i,j}^{n+1} - P_{i,j}^{n})$$

$$+ \frac{\Delta t}{(\phi h)_{i,j}} \{\frac{Q_{g_{i,j}} B_{g_{i,j}}}{\Delta x \Delta y} + \frac{B_{g_{i,j}}}{\Delta x^2} (\sigma_{xg_{i-\frac{1}{2},j}}$$

$$\{P_{i-1,j} - P_{i,j} + P_{cog_{i-1,j}} - P_{cog_{i,j}} - (\rho_g g)_{i-\frac{1}{2},j}$$

$$(z_{i-1,j} - z_{i,j})\} - \sigma_{xg_{i+\frac{1}{2},j}} \{P_{i,j} - P_{i+1,j}$$

$$+ P_{cog_{i,j}} - P_{cog_{i+1,j}} - (\rho_g g)_{i+\frac{1}{2},j} (z_{i,j} - z_{i+1,j})\})$$

$$+ \frac{B_{g_{i,j}}}{\Delta y^2} (\sigma_{yg_{i,j-\frac{1}{2}}} \{P_{i,j-1} - P_{i,j} + P_{cog_{i,j-1}}$$

$$- P_{cog_{i,j}} - (\rho_g g)_{i,j-\frac{1}{2}} (z_{i,j-1} - z_{i,j})\}$$

$$- \sigma_{yg_{i,j+\frac{1}{2}}} \{P_{i,j+1} - P_{i,j} + P_{cog_{i,j+1}} - P_{cog_{i,j}}$$

$$- (\rho_g g)_{i,j+\frac{1}{2}} (z_{i,j} - z_{i,j+1})\}) + \frac{B_{g_{i,j}}}{\Delta x^2}$$

$$(R_{s_{i-\frac{1}{2},j}} - R_{s_{i,j}})(\sigma_{xo_{i-\frac{1}{2},j}} \{P_{i-1,} - P_{i,j}$$

$$-(\rho_o g)_{i-\frac{1}{2},j} (z_{i-1,j} - z_{i,j})\}) - \frac{B_{g_{i,j}}}{\Delta x^2} (R_{s_{i+\frac{1}{2},j}} - R_{s_{i,j}})$$

$$(\sigma_{xo_{i+\frac{1}{2},j}}\{P_{i,j} - P_{i+1,j} - (\rho_o g)_{i+\frac{1}{2},j}\,(z_{i,j} - z_{i+1,j})\})$$

$$+ \frac{B_{g_{i,j}}}{\Delta y^2}\,(R_{s_{i,j-\frac{1}{2}}} - R_{s_{i,j}})(\sigma_{yo_{i,j-\frac{1}{2}}}\{P_{i,j-1} - P_{i,j}$$

$$- (\rho_o g)_{i,j-\frac{1}{2}}\,(z_{i,j-1} - z_{i,j})\}) - \frac{B_{g_{i,j}}}{\Delta y^2}\,(R_{s_{i,j+\frac{1}{2}}} - R_{s_{i,j}})$$

$$(\sigma_{xo_{i,j+\frac{1}{2}}}\{P_{i,j} - P_{i,j+1} - (\rho_o g)_{i,j+\frac{1}{2}}\,(z_{i,j+1} - z_{i,j})\})\}).$$

(C-4)

APPENDIX D

DESCRIPTION OF SIMULATION METHODS

In this appendix a brief description of each method used in this investigation to solve the pressure equation (C-1) will be given.

1. Alternating Direction Implicit Procedure

In the X-direction:

$$AX_{i,j}\, P^{n+\frac{1}{2}}_{i-1,j} + (-AX_{i,j} - CX_{i,j} - 2TRM_{i,j})P^{n+\frac{1}{2}}_{i,j}$$

$$+ CX_{i,j}\, P^{n+\frac{1}{2}}_{i+1,j} = (QTERM + PCTERM + GTERM)_{i,j}$$

$$- AY_{i,j}\, P^{n}_{i,j-1} - CY_{i,j}\, P^{n}_{i,j+1} - (-AY_{i,j} - CY_{i,j})P^{n}_{i,j}$$

$$- 2TRM_{i,j}P^{n}_{i,j}. \qquad \text{(D-1)}$$

In the Y-direction:

$$AY_{i,j}\, P^{n+1}_{i,j-1} + (-AY_{i,j} - CY_{i,j} - 2TRM_{i,j})\, P^{n+1}_{i,j} + CY_{i,j}\, P^{n+1}_{i,j+1}$$

$$= (QTERM + PCTERM + GTERM)_{i,j} - AX_{i,j}\, P^{n+\frac{1}{2}}_{i-1,j} - CX_{i,j}\, P^{n+\frac{1}{2}}_{i+1,j}$$

$$- (-AX_{i,j} - CX_{i,j})\, P^{n+\frac{1}{2}}_{i,j} - 2TRM_{i,j}\, P^{n+\frac{1}{2}}_{i,j}. \qquad \text{(D-2)}$$

The systems of equations (a set for each row) represented by (D-1) are solved first by a tridiagonal method of solution, where $P^{n+\frac{1}{2}}$ is an intermediate pressure approximation. Next the systems represented by

(D-2) are solved for P^{n+1}.

2. Alternating Direction Explicit Procedure

$$AX_{i,j}(P^{n+1}_{i-1,j} - P^{n+1}_{i,j}) + AY_{i,j}\,(P^{n+1}_{i,j-1} - P^{n+1}_{i,j})$$

$$+ CX_{i,j}(P^{n}_{i+1,j} - P^{n}_{i,j}) + CY_{i,j}\,(P^{n}_{i,j+1} - P^{n}_{i,j})$$

$$= (QTERM + PCTERM + GTERM)_{i,j} + \frac{TRM_{i,j}}{\Delta t}\,(P^{n+1}_{i,j} - P^{n}_{i,j}). \qquad (D\text{-}3)$$

Equation (D-3) can be solved explicitly at each grid point for $P^{n+1}_{i,j}$ since the pressure at (i-1,j) and (i,j-1) are given for the first row and first column by boundary conditions, and by the previous calculations as the successive points are evaluated.

3. Line Successive Over-relaxation Method

$$AX_{i,j}\,P^{*}_{i-1,j} + (-AX_{i,j} - CX_{i,j} - AY_{i,j} - CY_{i,j})\,P^{*}_{i,j}$$

$$+ CX_{i,j}\,P^{*}_{i+1,j} = (QTERM + PCTERM + GTERM)_{i,j} - AY_{i,j}\,P^{k}_{i,j-1}$$

$$- CY_{i,j}\,P^{k-1}_{i,j+1} + TRM_{i,j}\,(P^{*}_{i,j} - P^{n}_{i,j}). \qquad (D\text{-}4)$$

At iteration level k, (D-4) is solved for the jth row for P^{*} by a tridiagonal solution technique. This is possible, because at iteration level k, $AY_{i,j} = 0$ for the first row (J=1) and $P^{k-1}_{i,j+1}$ is known. (At the first iteration level $P^{k-1}_{i,j+1}$ is $P^{n}_{i,j+1}$).

Then $P^{k}_{i,j}$ is solved from the following equation:

$$P^{k}_{i,j} = w\, P^{*}_{i,j} + (1 - w)\, P^{k-1}_{i,j}. \qquad \text{(D-5)}$$

Proceeding to the next row (J=2), $P^{k}_{i,j}$ solved by (D-5) becomes $P^{k}_{i,j-1}$. Therefore, (D-4) can be solved for $P^{*}_{i,j}$ for J = 2. This calculation proceeds until all rows are computed.

The iteration parameter w is calculated by

$$w = \frac{2}{1 + \sqrt{1 - \rho(1)}}, \qquad \text{(D-6)}$$

where $\rho(1)$ is the spectral radius of the coefficient matrix, which is defined as:

$$\rho(1) = \left(\frac{\theta}{1 + \theta - \cos \frac{\pi}{M}} \right)^2, \qquad \text{(D-7)}$$

where θ is the ratio of the X-direction and Y-direction transmissibilities. The value of w is between 1 and 2.

4. Two-Direction Correction (2DC) to the Line Over-Relaxation Method

This is a method to calculate a correction factor to LSOR, in order to eliminate the highest eigenvalue of the coefficient matrix, so that LSOR will converge faster. The method is based on forcing the directional residuals in both X-direction and Y-direction to zero. If α is the X-direction correction factor, then the corrected pressure at iteration level k will be

$$P^{kc}_{i,j} = P^{k}_{i,j} + \alpha_j, \qquad \text{(D-8)}$$

where α_j is obtained by

$$\sum_i AY_{i,j}\,\alpha_{j-1} + \left(\sum_i(-AY_{i,j} - CY_{i,j}) - \sum_i TRM_{i,j}\right)\alpha_j + \sum_i CY_{i,j}\alpha_{j+1} = \sum_i R_{i,j}. \qquad \text{(D-9)}$$

Equation (D-9) yields a tridiagonal coefficient matrix which can be solved for α_j for all j's.

Similarly, for the Y-direction a parameter β_i, which is analogous to α_j can be calculated for all i's, and the corrected pressure will be

$$P^{kc}_{i,j} = P^{k}_{i,j} + \alpha_j + \beta_i .$$

5. Strongly Implicit Procedure

This technique is one of the better studied. The following equation is given as a basis for discussing this method.

$$AX_{i,j}\,P_{i-1,j} + AY_{i,j}\,P_{i,j-1} + BB_{i,j}\,P_{i,j} + CY_{i,j}\,P_{i,j+1} + CX_{i,j}\,P_{i+1,j} = DX_{i,j} . \qquad \text{(D-10)}$$

In matrix form equation (D-10) can be written as

$$AP = D, \qquad \text{(D-11)}$$

where A is the matrix of coefficient, P is the solution vector, and D is a vector of known parameters in the equations. In order to make it amenable to direct solution, matrix A is altered into (A + B), so that when (A + B) is factored into a product of a lower matrix, L, and an

upper matrix, U, L and U have only three non-zero elements. The diagonal elements of L will then be (d, c, b) and the diagonal elements of U will be (1, e, f).

The altered matrix (A + B) has then become a seven-diagonal matrix. The pressures associated with the two additional elements were chosen to be those occupying the grid points (i+1, j-1) and (i-1, j+1), and they are calculated by using the Taylor's series expansion with higher order terms dropped.

$$P_{i+1,j-1} = -P_{i,j} + P_{i+1,j} + P_{i,j-1}, \qquad \text{(D-12)}$$

$$P_{i-1,j+1} = -P_{i,j} + P_{i-1,j} + P_{i,j+1}. \qquad \text{(D-13)}$$

The altered matrix (A + B) is made compatible with the original matrix A by assigning an iteration parameter α to the right hand sides of the equations (D-12) and (D-13) and subtracting them from the corresponding left hand sides of equations (D-12) and (D-13). The unknown side of equation (D-10) becomes

$$AX_{i,j} P_{i-1,j} + AY_{i,j} P_{i,j-1} + BB_{i,j} P_{i,j} + CY_{i,j} P_{i,j+1}$$

$$+ CX_{i,j} P_{i+1,j} + E_{i,j}\{P_{i+1,j-1} - \alpha(-P_{i,j}$$

$$+ P_{i+1,j} + P_{i,j-1})\} + F_{i,j}\{P_{i-1,j+1} - \alpha(-P_{i,j}$$

$$+ P_{i-1,j} + P_{i,j+1})\},$$

where $E_{i,j} = b_{i,j}\, e_{i,j-1}$,

and $F_{i,j} = c_{i,j}\, f_{i-1,j}$.

The additional relations are:

$$b_{i,j} = AY_{i,j} - \alpha E_{i,j},$$

$$c_{i,j} = AX_{i,j} - \alpha F_{i,j},$$

$$d_{i,j} + b_{i,j} f_{i,j-1} + c_{i,j} e_{i-1,j} = BB_{i,j} + \alpha E_{i,j} + \alpha F_{i,j},$$

$$d_{i,j} e_{i,j} = CX_{i,j} - \alpha E_{i,j},$$

$$d_{i,j} f_{i,j} = CY_{i,j} - \alpha F_{i,j}.$$

The iterative method is derived by adding the product BP to both sides and (AP - AP) to the right hand side of equation (D-10), i.e.,

$$(A + B)P^{n+1} = (A + B)P^{n} - (AP^{n} - D). \qquad \text{(D-14)}$$

The equation (D-14) is solved as follows:

Set $\delta^{n+1} = P^{n+1} - P^{n}$,

and calculate

$$R^{n} = D - AP^{n}.$$

The equation (D-14) can then be written as

$$(A + B)\delta^{n+1} = R^{n}. \qquad \text{(D-15)}$$

Since (A + B) = LU, then equation (D-15) can be written as

$$LU\delta^{n+1} = R^{n}.$$

Substituting V for $U\delta^{n+1}$, equation (D-15) becomes

$$LV = R^n.$$

Therefore equation (D-15) can be solved by a tridiagonal solution technique.

The iteration parameters are set between 0.0 and 1.00, and the values are calculated by the following formula:

$$(1 - \alpha_{max}) = \min. \left(\frac{2\Delta x^2}{1 + \frac{KY\Delta x^2}{KX\Delta y^2}}, \frac{2\Delta y^2}{1 + \frac{KX\Delta y^2}{KY\Delta x^2}}\right)^*. \qquad \text{(D-16)}$$

Testing showed SIP to be a powerful solution technique comparable to the approach developed in this investigation.

KX is the transmissibility in the X-direction, and KY is the transmissibility in the Y-direction.

APPENDIX E

NOMENCLATURE

A	cross sectional area - cm^2
B	formation volume factor - res cm^3/std cm^3
C	compressibility factor - vol/vol/atm
g	conversion factor for the gravity term
h	net pay thickness - cm
I	counting number in the X-direction
J	counting number in the Y-direction
k	absolute permeability - Darcy
k_{eff}	effective permeability - Darcy
k_r	relative permeability - fraction
M	number of blocks in the X-direction
N	number of blocks in the Y-direction
P	pressure - atm
PV	pore volume - cm^3
Q	production or injection rate - cm^3/sec
R_s	solution gas - oil ratio - std cm^3 of gas/std cm^3 of oil
S	saturation - fraction
t	time-second
w	iteration parameter for LSOR
z	subsea depth - cm
α	pressure correction factor to LSOR in the X-direction - atm
α	iteration parameter for SIP
β	pressure correction factor to LSOR in the Y-direction - atm

Δ	increment
ϕ	porosity - fraction
Φ	pressure potential - atm
μ	viscosity - cp
ρ	density - gm/cm^3

Subscripts:

abs	absolute
BP	bubble-point
g	gas phase
i	position in the X-direction
IN	influx
j	position in the Y-direction
max	maximum
min	minimum
o	oil phase
og	oil-gas interface
or	residual oil
OUT	efflux
ow	oil-water interface
pv	pore volume
r	relative
wc	irreducible water
x	in the X-direction
y	in the Y-direction

Superscript:

k	current iteration level
k+1	next iteration level
n	current time level
n-1	previous time level
n+1	next time level

237286